Econ 172A - Slides from Lecture 11

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Announcements

- Midterm:
  Graded midterms available in Section on Nov. 6 and in class on Nov. 8.
  Please consult posted answers and regrading instructions.
- Quiz 3 on Thursday, November 15. End of Class.
- Topic: Sensitivity.
- Problems on Branch and Bound: 2007: Homework 3, #1; Final: #5,6 2008: Homework 3 # 1, 2; Final: #4.

Example

$$
\begin{array}{rrcrcrcrcrl}
\max & 2x_1 & + & 4x_2 & + & 3x_3 & + & x_4 & & & (0) \\
\text{subject to} & 3x_1 & + & x_2 & + & 4x_3 & + & x_4 & \leq & 3 & (1) \\
 & x_1 & - & 3x_2 & + & 2x_3 & + & 3x_4 & \leq & 3 & (2) \\
 & 2x_1 & + & x_2 & + & 3x_3 & - & x_4 & \leq & 6 & (3)
\end{array}
$$

with the additional constraint that each variable be 0 or 1.

Bound

The first step of the procedure is to find an upper bound for the value of the problem.
Standard method: Solve the "relaxed" problem (without integer constraints).

Relaxation

$$
\begin{array}{lrcrcrcrcrl}
\text{max} & 2x_1 & + & 4x_2 & + & 3x_3 & + & x_4 & & & (0) \\
\text{subject to} & 3x_1 & + & x_2 & + & 4x_3 & + & x_4 & \leq & 3 & (1) \\
 & x_1 & - & 3x_2 & + & 2x_3 & + & 3x_4 & \leq & 3 & (2) \\
 & 2x_1 & + & x_2 & + & 3x_3 & - & x_4 & \leq & 6 & (3)
\end{array}
$$

and $x \geq 0$.

- Same as before, except integer constraints are gone.
- Value of relaxed problem must be greater than or equal to value of original problem. (Why?)
- Value of relaxed problem is equal to value of original problem exactly when related problem has an integer solution.
- Relaxed problem should be easier to solve than original problem.

Solution to Relaxed Problem

- $x = (x_1, \ldots, x_4) = (0, 1, 1/4, 1)$ with value 23/4.
- You can check this (using complementary slackness).
- Finding the solution requires Excel.
- So value of the original problem is is no more than 23/4.
- In fact, since the value of the original problem must be an integer, the value of the original problem can be at most 5.

What Next?

- The value of the problem cannot be greater than 5.
- If we could figure out a way to make the value equal to 5 we would be done.
- A guess: Set $x_3 = 0$ (instead of $1/4$). Take $x_2 = x_4 = 1$ and $x_1 = 0$ (as before).
- This is feasible for the original integer program and gives value 5.
- Hence it is a solution.

If you are able to make the guess, then you could quit.

Branch and Bound works even if you cannot guess.

Branch

- Assume that you didn't find a feasible way to attain the upper bound of 5.
- The original problem has four variables.
- It would be easier to solve it if had three variables.
- Form two related subproblems involving three variables.

Branching

- Observation: When you solve the problem, either $x_1 = 0$ or $x_1 = 1$.
- So if I can solve two subproblems (one with $x_1 = 0$ and the other with $x_1 = 1$), then I can solve the original problem.
- I obtain subproblem I by setting $x_1 = 0$. This leads to

$$\begin{array}{lrcrcrcrr} \text{max} & 4x_2 & + & 3x_3 & + & x_4 & & & (0) \\ \text{subject to} & x_2 & + & 4x_3 & + & x_4 & \leq & 3 & (1) \\ & -\ 3x_2 & + & 2x_3 & + & 3x_4 & \leq & 3 & (2) \\ & x_2 & + & 3x_3 & - & x_4 & \leq & 6 & (3) \end{array}$$

- This problem is the original problem with $x_1 = 0$.

The other subproblem

Subproblem II, comes from setting $x_1 = 1$.

$$\begin{array}{lrcrcrcrl}
\text{max} & 4x_2 & + & 3x_3 & + & x_4 & & & (0) \\
\text{subject to} & x_2 & + & 4x_3 & + & x_4 & \leq & 0 & (1) \\
 & -\ 3x_2 & + & 2x_3 & + & 3x_4 & \leq & 2 & (2) \\
 & x_2 & + & 3x_3 & - & x_4 & \leq & 4 & (3)
\end{array}$$

This constitutes the "branching" part of the branch and bound method. Now comes to the bounding part again.

Bounding Again

- Bound Subproblem I.
- Solve Relaxed Problem I.
- It is the same as Original Relaxed Solution because $x = (0, 1, 1/4, 1)$, the solution to original relaxed problem is feasible for relaxed subproblem I.
- Relaxed subproblem II has solution $(1, 0, 0, 0)$. [Use excel or common sense.]
- Since $(1, 0, 0, 0)$ is integer, it is feasible for subproblem II.
- So we have solved Subproblem II. It has value 2

What we now know

1. Upper bound is 5.
2. Lower bound is 2.
3. We know everything about Subproblem II.
4. Subproblem I is "easier" than original problem (only three variables).
5. Repeat procedure on Subproblem I (branching).

Branching Again

- Break Subproblem I into two subproblems.
- Subproblem I.I in which $x_2 = 0$ (and $x_1 = 0$).
- Subproblem I.II in which $x_2 = 1$ (and $x_1 = 0$).
- At this point the remaining problems are probably easy enough to solve by observation: $x_3 = 0$ and $x_4 = 1$ for Subproblem I.I. (This means that the possible solution identified by solving subproblem I.I is $(0, 0, 0, 1)$ with value 1.)
- For Subprogram I.II the solution is also $x_3 = 0$ and $x_4 = 1$, but to get to this subproblem we set $x_2 = 1$, so the possible solution identified from this computation is $(0, 1, 0, 1)$ with value 5. (If you do not see how I obtained the values for x_3 and x_4, then carry out the branching step one more time.)

Assessment

At this point, we have the following information:

1. The value of the problem is no more than 5.
2. There are three relevant subproblems.
3. The value of Subproblem II is 2.
4. The value of Subproblem I.I is 2.
5. The value of Subproblem I.II is 5.

Consequently, Subproblem I.II really does provide the solution to the original problem.

Recap

- This exercise illustrate the branch-and-bound technique, but it does not describe all of the complexity that may arise.
- I will describe the technique in general terms.
- Then another example.

Branch and Bound: Context

- Given an integer programming maximization problem with n variables in which all variables can take on the values 0 or 1.
- Generalization is easy.

Branch and Bound Rules

1. Set $\underline{v} = -\infty$.
2. Bound the original problem. To do this, you solve the problem ignoring the integer constraints and round the value down to the nearest integer.
3. If the solution to the relaxed problem in Step 2 satisfies the integer constraints, stop. You have a solution to the original problem. Otherwise, call the original problem a "remaining subproblem" and go to Step 4.
4. Among all remaining subproblems, select the one created most recently. If more than one has been created most recently, pick the one with the larger bound. If they have the same bound, pick randomly. Branch from this subproblem to create two new subproblems by fixing the value of the next available variable to either 0 or 1.

Continued

5. For each new subproblem, obtain its bound z by solving a relaxed version and rounding the value down to the nearest integer (if the relaxed solution is not an integer).
6. Attempt to fathom each new subproblem.

Fathoming

Fathoming is completing the analysis of a subproblem.

You can do this in three ways.

1. A problem is fathomed if its relaxation is not feasible.
2. A problem is fathomed if its value is less than or equal to $\underline{v}$.
3. A problem is fathomed if its relaxation has an integer solution.

All subproblems than are not fathomed are remaining subproblems.

Continuation of Description

7 If one of the subproblems is fathomed because its relaxation has an integer solution, update $\underline{v}$ by setting it equal to the largest of the old value of $\underline{v}$ and the value of the largest relaxation with an integer solution. Call a subproblem that attains $\underline{v}$ the candidate solution.

8 If there are no remaining subproblems, stop. The candidate solution is the true solution. (If you stop and there are no candidate solutions, then the original problem is not feasible.) If $\underline{v}$ is equal to the highest upper bound of all remaining subproblems, stop. The candidate solution is the true solution. Otherwise, return to Step 4.

Discussion

- Steps 1, 2, and 3 initialize the algorithm.
- You start by "guessing" that the value of the problem is $-\infty$.
- In general, $\underline{v}$ is your current best feasible value.
- Next you get an upper bound by ignoring integer constraints.
- If ignoring integer constraints does not lead to non-integer solutions, you are done.
- Otherwise, you move to Step 4 and try to solve smaller problems.
- In Step 4 you first figure out how to branch.
- Branching involves taking a subproblem that has yet to be solved or discarded and simplifying it by assigning a value to one of the variables.
- When variables can take on only the values 0 or 1, this creates two new problems.
- In Step 5 you find an upper bound to the value of these new problems by ignoring integer constraints.
- In Step 6 you try to "fathom" some of the new problems.

Fathoming Rules

1. You can fathom a problem if its relaxation is not feasible. If the relaxation is not feasible, then the problem itself is not feasible. Hence it cannot contain the solution to the original problem.
2. You can fathom the problem if its upper bound no higher than the current value of $\underline{v}$. In this case, the problem cannot do better than your current candidate solution.
3. You can fathom a problem if the relaxation has an integer solution. This case is different from the first two. In the first two cases, when you fathom a problem you discard it. In this case, when you fathom a problem you put it aside, but it is possible that it becomes the new candidate solution.

More Discussion

- In Step 7 you update your current best feasible value, taking into account information you have learned from problems you recently fathomed because you found their solutions (third option in Step 6).
- In Step 8 you check for optimality.
- If you have fathomed all of the problems, then you have looked at all possible solutions.
- It is not possible to do better than your current candidate.
- If you do not have a current candidate it is because you never managed to solve a subproblem. If you can eliminate all remaining subproblems without finding a solution, then the feasible set of the original problem must have been empty.
- If you have not fathomed all of the problems, then you return to Step 4 and try to do so.
- Since eventually you will assign values to all variables, the process must stop in a finite number of steps with a solution (or a proof that the problem is not feasible).

Example

Knapsack Problem.

- 6 items.
- Values: $v = (v_1, v_2, \ldots, v_6) = (1, 4, 9, 16, 25, 36)$
- Weights: $w = (w_1, w_2, \ldots, w_6) = (1, 2, 3, 7, 11, 15)$
- Capacity: 27.

Relaxed Solution

- If you solve the original problem without integer constraints you obtain: $(0, 0, 1, 1, 2/11, 1)$ with value 65 6/11.
- 65 (which is an integer) becomes your new upper bound.
- Since the relaxed solution is not a solution, you must continue to the branching step.

Branch

- The branching step generates two problems, one in which $x_1 = 0$ and the other in which $x_1 = 1$.
- In the first case, the upper bound is 65 as before.
- In the second case, the solution to the relaxed problem is $(1, 0, 1, 1, 1/11, 1)$ with value 64 3/11, which rounds down to 64.
- Unfortunately, you cannot fathom anything.

More Branching

- Two remaining subproblems.
- The one with the higher upper bound has $x_1 = 0$.
- Branch on this problem by setting $x_2 = 0$ and $x_2 = 1$.
- In the first case the solution to the problem is $(0, 0, 1, 1, 2/11, 1)$ with value 65.
- In the second case, the solution to the problem is $(0, 1, 1, 1, 0, 1)$ and has value 65.
- This problem is fathomed (because the solution to the relaxed problem is in integers).
- Update $\underline{v}$.
- I have a solution to the original problem because I attained the upper bound with a feasible vector.