Econ 172A - Slides from Lecture 10

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Announcements

- Important: Midterm seating assignments. Posted.
- Answers to Quiz 2 posted.
- Midterm on November 1, 2012.
- Problems for Midterm:
    1. 2004: Midterm #1: 2, 3; Midterm #2. Problem Set 1, Problem Set 2 - 1,2. Final: 2, 3, 5.
    2. 2007: Midterm. Problem Sets 1 and 2.
    3. 2008: Midterm. Problem Sets 1 and 2. Final: 5, 7 a, b.
    4. 2010: Midterm. Quizzes 1, 2. Final 1 a,b; 4

Material for Midterm

- Problem Formulation
- Graphing
- Duality (all topics)
- Interpreting Excel Spreadsheets
  I can ask you to "fill in the blanks" in a sensitivity table.

This means: material through last Thursday's lecture, but today's topic practices useful skills for the exam.

Example

Simple formulation exercise.

- A furniture company that makes tables and chairs.
- A table requires 40 board feet of wood.
- A chair requires 30 board feet of wood.
- Wood costs $1 per board foot and 40,000 board feet of wood are available.
- It takes 2 hours of labor to make an unfinished table or an unfinished chair.
- 3 more hours of labor will turn an unfinished table into a finished table.
- 2 more hours of labor will turn an unfinished chair into a finished chair.
- There are 6000 hours of labor available.
- No need to pay for this labor.

Prices

The prices of output are given in the table below:

Product	Price
Unfinished Table	$70
Finished Table	$140
Unfinished Chair	$60
Finished Chair	$110

Formulation

- Objective: Describe the production plans that the firm can use to maximize its profits.
- Variables: Number of finished and unfinished tables and chairs.
- Let T_F and T_U be the number of finished and unfinished tables.
- Let C_F and C_U be the number of finished and unfinished chairs.
- Revenue:

$$70T_U + 140T_F + 60C_U + 110C_F,$$

.

- Cost is $40T_U + 40T_F + 30C_U + 30C_F$ (because lumber costs \$1 per board foot).
- Profit (Revenue - Cost): $30T_U + 100T_F + 30C_U + 80C_F$.

- The constraints are:
  1. $40T_U + 40T_F + 30C_U + 30C_F \leq 40000$.
  2. $2T_U + 5T_F + 2C_U + 4C_F \leq 6000$.
- The first constraint says that the amount of lumber used is no more than what is available.
- The second constraint states that the amount of labor used is no more than what is available.

Solution

- Excel finds the answer to the problem to be to construct only finished chairs (1333.333).
- It is crazy to produce $\frac{1}{3}$ chair.
- This means that the LP formulation is not literally correct.
- Now: We pretend that fractional chairs are ok.
- Later: We will introduce methods to deal with the constraint that variables should be integers.
- The profit is $106,666.67.

Sensitivity questions

- What would happen if the price of unfinished chairs went up?
  - Currently they sell for $60.
  - Allowable increase in the coefficient is $50, it would not be profitable to produce them even if they sold for the same amount as finished chairs.
  - If the price of unfinished chairs went down, then certainly you wouldn't change your solution.
- What would happen if the price of unfinished tables went up?
  - The allowable increase is greater than 70.
  - Even if you could sell unfinished tables for more than finished tables, you would not want to sell them.
  - How could this be?
  - At current prices you don't want to sell finished tables.
  - Hence it is not enough to make unfinished tables more profitable than finished tables, you must make them more profitable than finished chairs.
  - Doing so requires an even greater increase in the price.

More Questions

- What if the price of finished chairs fell to \$100?
  - This change would alter your production plan: it involves a \$10 decrease in the price of finished chairs and the allowable decrease is only \$5.
  - Complete answer: resolve problem.
  - The best thing to do is specialize in finished tables, producing 1000 and earning \$100,000.
  - If you continued with the old production plan your profit would be $70 \times 1333\frac{1}{3} = 93,333\frac{1}{3}$, so the change in production plan was worth more than \$6,000.

- How would profit change if lumber supplies changed?
  - The shadow price of the lumber constraint is \$2.67.
  - The range of values for which the basis remains unchanged is 0 to 45,000.
  - This means that if the lumber supply went up by 5000, then you would continue to specialize in finished chairs, and your profit would go up by $\$2.67 \times 5000 = \$10,333$.
  - If lumber supply increased by more, you run out of labor and want to reoptimize.
  - If lumber supply decreased, then your profit would decrease, but you would still specialize in finished chairs.

- How much would you be willing to pay an additional carpenter?
  - Skilled labor is not worth anything to you.
  - You are not using the labor than you have.
  - You would pay nothing for additional workers.
- Suppose that industrial regulations complicate the finishing process, so that it takes one extra hour per chair or table to turn an unfinished product into a finished one. How would this change your plans? (This problem differs from the original one because the amount of labor to create a finished product increases by one unit.)
  - You cannot read your answer off the sensitivity table, but a bit of common sense tells you something.
  - The change cannot make you better off.
  - To produce 1,333.33 finished chairs you'll need 1,333.33 extra hours of labor.
  - You do not have that available.
  - So the change will change your profit.
  - Using Excel, it turns out that it becomes optimal to specialize in finished tables, producing 1000 of them and earning $100,000.

New Activity

- The owner of the firm comes up with a design for a beautiful hand-crafted cabinet.
- Each cabinet requires 250 hours of labor (this is 6 weeks of full time work) and uses 50 board feet of lumber.
- Suppose that the company can sell a cabinet for $200, would it be worthwhile?

One Approach

- Change problem by adding an additional variable and an additional constraint.
- The coefficient of cabinets in the objective function is 150, which reflects the sale price minus the cost of lumber.
- The final value increases to 106,802.7211.
- The solution involved reducing the output of unfinished chairs to 1319.727891 and increasing the output of cabinets to 8.163265306.
- You could not have guessed these figures in advance, but you could figure out that making cabinets was a good idea.

Using Sensitivity Analysis

- Value the inputs to the production of cabinets.
- Cabinets require labor, but labor has a shadow price of zero.
- They also require lumber.
- The shadow price of lumber is \$2.67, which means that each unit of lumber adds \$2.67 to profit.
- Hence 50 board feet of lumber would reduce profit by \$133.50.
- Since this is less than the price at which you can sell cabinets (minus the cost of lumber), you are better off using your resources to build cabinets.
- You can check that the increase in profit associated with making cabinets is \$16.50, the added profit per unit, times the number of cabinets that you actually produce.
- If the price of cabinets was \$150 the additional option does not lead to cabinet production.

Integer Programming

- LP assumes that everything is “perfectly divisible.”
- Half a chair?
- Half a car?
- We would like to be able to talk about situations in which things are constrained to be whole numbers.

Standard Model

1. $c = (c_1, \dots, c_n)$ objective function coefficients.
2. $b = (b_1, \dots, b_m)$ right-hand sides.
3. A a matrix with m rows and n columns (and entry a_{ij} in row i and column j) (technology)
4. $\mathcal{I}$ a subset of $\{1, \dots, n\}$

Only (4) is new.
Find $x = (x_1, \dots, x_n)$

$$\max c \cdot x \text{ subject to } Ax \leq b, x \geq 0, x_j \text{ is an integer whenever } j \in \mathcal{I}. \tag{1}$$

Novelty

The set $\mathcal{I}$ and the constraint that x_j is an integer when $j \in \mathcal{I}$.

Integer Linear Programming

1. $\mathcal{I}$ empty – standard LP.
2. $\mathcal{I}$ nonempty but not everything "mixed integer programming".
3. $\mathcal{I}$ everything – our topic.

Observations

- Adding constraints lowers your value (or the value stays the same). It can't go up.
- No solution algorithms that are sure to work quickly.
- People study specialized models.
- People introduce solution procedures that are not guaranteed to solve the problem efficiently, but are still useful.

Examples

[The Transportation Problem] Given a finite number of suppliers, each with fixed capacity, a finite number of demand centers, each with a given demand, and costs of transporting a unit from a supplier to a demand center, find the minimum cost method of meeting all of the demands without exceeding supplies.

[Assignment Problem] Given equal numbers of people and jobs and the value of assigning any given person to any given job, find the job assignment (each person is assigned to a different job) that maximizes the total value.

[Shortest Route Problem] Given a collection of locations the distance between each pair of locations, find the cheapest way to get from one location to another.

[Maximum Flow Problem] Given a series of locations connected by pipelines of fixed capacity and two special locations (an initial location or source and a final location or sink), find the way to send the maximum amount from source to sink without violating capacity constraints.

[Knapsack Problem] Given a knapsack with fixed capacity and a collection of items, each with a weight and value, find the items to put in the knapsack that maximizes the total value carried subject to the requirement that that weight limitation not be exceed.

Knapsack Formulation

- Data:
  1. N, (integer) number of items
  2. C, the capacity of the knapsack
  3. For each $j = 1, \ldots, N$, the value $v_j > 0$
  4. and weight $w_j > 0$ of item j.
- Figure out which items to pack to maximize value carried subject to capacity constraint.

Trick

Let x_j be a variable that is equal to 1 if you place item j in the knapsack and equal to 0 if you do not.

This is a clever way to define variables.

x_j "counts" whether you take item j or not.

Given this definition, writing constraints and objective function is easy (or at least easier).

Expressions

If $x = (x_1, \ldots, x_N)$ and all of the x_j take on the value 0 or 1, then

$$\sum_{j=1}^{N} v_j x_j$$

is equal to the value of what you put in the knapsack and

$$\sum_{j=1}^{N} w_j x_j$$

is equal to the weight of what you put in the knapsack.

Formulation

$$\max \sum_{j=1}^{N} v_j x_j$$

subject to

$$\sum_{j=1}^{N} w_j x_j \leq C,$$

$$0 \leq x_j \leq 1,$$

and x_j integer.

In this formulation, except for the restriction that x_j takes an integer values, the constraints are linear. Notice that by requiring that x_j both be between 0 and 1 and be an integer, we are requiring that x_j be either zero or 1.