Econ 172A, Fall 2010: Final Examination Answers

Grading Notes Exam: 400 points possible, high: 375 , low: 118, median: 257.

Class: 800 points possible (final + two times midterm score + three best quizzes), high: 731, low: 222, median: 520. Grade cutoffs: A $\geq$ 620; B $\geq$ 531; C $\geq$ 377. (I gave plusses and minuses in marginal cases.)

1. 70 points total. Part a: 20, b: 20, c: 30. Many people got the right numerical answer to (b) but gave no explanation (or an incorrect one). They received 10 points (or more, depending on the quality of the explanation). This happened to a lesser extent on part (c) as well.

2. 70 points: part (a): 25 points, (b): 5 points, (c): 20 points; (d) 10 points. Full credit for correct answers that don't use algorithm but provide accurate and true justifications, but little partial credit for incorrect answers and no description of the computation.

3. 40 points: 15 points for coming up with a feasible shipping schedule (there are many possible, but it should not be hard to identify which are feasible); 5 points for correctly computing the cost; 20 points for the second part. We gave 15 points for saying "this is an assignment problem (or a transportation problem)" and nothing providing an explanation. We gave partial credit for other (incorrect) answers. I discussed a problem like this on the last day of class.

4. 60 points: Assuming that they use my variable definitions, give 10 points for the objective function, 15 points for each of the substantive constraints and 5 points for non-negativity. You may give partial credit if constraints "almost" make sense.

5. 50 points (ten points for each part)

6. 60 points (ten points for each part)

7. 50 points (part (a) - 5 points; part (b) - 10 points; part (c) – 10 points; part (d) - 25 points: 3 points for each correct non-zero number in rows 2-6, 1 point for each non-zero number in row 1, .5 points for each NEI (rounded up, total score should be an integer).

1. Consider the problem: Find x_1, x_2, and x_3 to solve:

$$\min 2x_1 - x_2 + x_3$$

subject to

$$7x_1 + 5x_2 + 10x_3 \geq 8.$$

(a) Solve the problem (any method) subject to the (additional) constraint that $x = (x_1, x_2, x_3) \geq 0$.

One way to solve the problem is to write the dual, $\max 8y$ subject to $7y \leq 2, 5y \leq -1, 10y \leq 1, y \geq 0$, observe that this is infeasible (cannot have $5y \leq -1$ and $y \geq 0$), and that primal is feasible (all variables equal to one, for example). This implies that problem is unbounded by duality theorem. Alternatively, notice that you can make x_2 arbitrarily large. This will satisfy the constraints (as long as x_1 and x_3 are non-negative) and make objective function arbitrarily large. More formally: if $(x_1, x_2, x_3) = (0, M, 0)$ for $M > 1.6$ you satisfy the constraints and make objective function value $-M$, so by increasing M you can make the value of the objective function lower than any number.

(b) Solve the problem (any method) subject to the (additional) constraint that $1 \geq x_i \geq 0$ for $i = 1, 2, 3$.

Since objective function is decreasing in x_2 and constraint becomes easier to satisfy as x_2 increases, you want $x_2 = 1$ in the solution. The remaining problem:

$$\min 2x_1 + x_3$$

subject to

$$7x_1 + 10x_3 \geq 3, 0 \leq x_1, x_3 \leq 1$$

you can solve graphically or by analogy with the (relaxed) knapsack problem: $x_1 = 0$ and $x_3 = .3$. Solution: $(x_1, x_2, x_3) = (0, 1, .3)$ value: -.7

(c) Solve the problem (any method) subject to the (additional) constraint that $x_i \in \{0, 1\}$ for $i = 1, 2, 3$ (that is, the variables can take only the values zero and one).

Here you can note that there are only 8 possible choices (since the variables can take on only the values 0 and 1) and check all possibilities. You can note (as in the previous part) that $x_2 = 1$ and then note that there are only four simple cases to check. You can branch and bound. All procedures lead to $(x_1, x_2, x_3) = (0, 1, 1)$ value: 0

2. (a) Find a maximum flow for this network. You may describe the flow here or on one of the network diagrams on the next page. Please indicate your answer clearly. You must explain how you found the answer. If you properly used the algorithm introduced in class (and you show the steps), then you need no further justification. If you used another method, then you must explain the method and explain how it works.

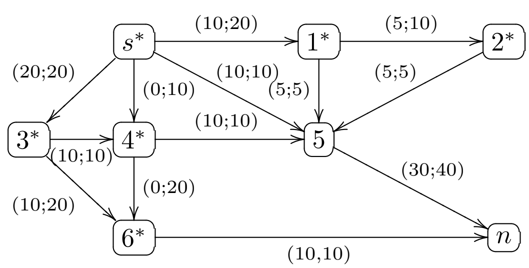
I describe the maximum flow in the diagram below. I used the class algorithm (four steps: 5 on $(s, 1, 5, n)$; 5 on $(s, 1, 2, 5, n)$, 10 on $(s, 3, 4, 5, n)$; 10 on $(s, 5, n)$ and 10 on $(s, 3, 6, n)$).

(b) Find the value of the maximum flow. 40

(c) Find a minimum capacity cut for this network. The indicated flow is maximal (the first number on each arc is the flow). The starred nodes describe the last step of the algorithm. Notice that you need to use the "reverse flow" part of the algorithm to label node 3. The minimum capacity cut is $\{s, 1, 2, 3, 4, 6\}$ and $\{5, n\}$. It has capacity 40.

(d) What is the capacity of the minimum capacity cut? 40.

(e) Find the capacity of the cut $\{s, 3\}$ and $\{1, 2, 4, 5, 6, n\}$. The capacity of the cut $\{s, 3\}$ and $\{1, 2, 4, 5, 6, n\}$ is 70.



3. Byclops manufactures bicycles at factories in Portland, San Francisco, and Bakersfield. The bikes are sent to regional distributors in Chicago, St. Louis, and Atlanta. The shipping costs vary. The company would like to find the least-cost way to meet the demands at each of the distribution centers. Chicago needs to receive 800 bikes each month, St. Louis needs 600, and Atlanta needs 200. Portland produces 850 each month, San Francisco 650, and Bakersfield 300. The shipping cost per unit from each pair of cities is given in the table below:

	Chicago	St. Louis	Atlanta
Portland	8	12	10
San Francisco	10	14	9
Bakersfield	11	8	12

It costs nothing for Byclops to leave a bicycle at the factory (rather than ship it to a distribution center).

(a) Is it possible for Byclops to meet the given demands with its available supplies? If it is, then exhibit a feasible method to do this and compute the cost of the method.

Yes. There is more total supply ($850 + 650 + 300 = 1800$) than demand ($800+600+200 = 1600$). There are many ways to meet the demand. One way is:

i. Ship 800 bikes to Chicago from Portland (cost 8*800 = 6400)
ii. Ship 600 bikes to Atlanta from San Francisco (cost 14*600 = 8400)
iii. Ship 200 bikes to Atlanta from Bakersfield (cost 12*200 = 2400)
iv. total cost $6400 + 8400 + 2400 = 17200$

Note: this is only one of many possible correct answers.

(b) Explain which algorithm presented in the class could be used to solve this problem and why. Describe briefly how you would set up the problem in order to apply the algorithm. You do not need to solve the problem. It is sufficient to explain in two or three sentences how you set up the problem. Your explanation must be enough to explain how the procedure would produce the answer.

You can think of this as an assignment problem. You need to imagine that there are 1800 bicycles (850 from Portland, 650 from San Francisco, and 300 from Bakersfield) and 1800 demand locations (1600 from the cities and an extra 200 for the surplus). The costs come from the table: It costs 0 to take a bike and "transport it" to the surplus location. Otherwise, the table tells you the costs. Now solve this (big) problem using the algorithm for solving assignment problems.

4. objective function: $\max 11x_A + 23x_B - 5X_R$

   Constraints:

   Assembly line 1 constraint: $2x_A + 2x_B + 2x_P \leq 800$

   Assembly line 2 constraint: $2x_B \leq 600$

   Raw material: $x_A + 3x_B - x_R - x_P \leq 0$

   All variables non-negative.

5. For each of the statements below indicate whether the statement is always **TRUE**, by writing "TRUE" otherwise write "FALSE." No justification is required.

   This question refers to a network in which there are N nodes and in which $c(i, j)$ is the cost of going from node i to node j and all pairs of nodes are connected. Assume that each pair of nodes is connected, the costs are finite, positive integers.

Some parts use the following terms:

The costs are **distinct** if $c(i,j) = c(i',j')$ if and only if $i = i'$ and $j = j'$. Costs are distinct in Parts (a) and (b) (but costs may or may not be distinct in Parts (c), (d), and (e)).

A **cheapest edge** is an edge in which $c(i,j)$ is the smallest.

A **most expensive edge** is an edge in which $c(i,j)$ is the largest.

(a) If the costs are distinct, then the minimum spanning tree does not contain a most expensive edge. False. If the tree had only two nodes (and one edge), then the most expensive and the least expensive edge are the same. (The answer would be true if there were more than two nodes. One can then start the algorithm at one of the nodes with the most expensive edge attached. The algorithm would never pick this edge.)

(b) If the costs are distinct, then the minimum spanning tree is unique.

True. At each step, the algorithm can pick one and only one thing.

(c) A cheapest edge is always part of the minimum spanning tree.

False. All edges could cost the same amount. In which case, the MST would not include all cheapest edges.

(d) There always exists a minimum spanning tree that contains a cheapest edge.

True. You may start by including a cheapest edge.

(e) The value of the minimum spanning tree must be at least N.

False. It could be $N - 1$ (if all costs are one).

6. For each of the statements below indicate whether the statement is always **TRUE**, by writing "TRUE" otherwise write "FALSE." No justification is required.

This question refers to an assignment problem in which there are N people and N jobs. The benefit of assigning person i to job j is $a(i,j)$. For each $i, j = 1, \ldots N$, $a(i,j)$ is a positive integer. A feasible assignment associates each person with one job and each job with one person. A solution to the assignment problem is a feasible assignment that generates the largest total benefit. The value of an assignment problem is the total benefit generated by the solution.

(a) The assignment problem has $N!$ feasible assignments.

True. The first person can be assigned to any of the N jobs. The second person can be assigned to any of the remaining $N - 1$ jobs. And so on.

(b) The solution to the assignment problem will also solve the assignment problem obtained by multiplying all benefits by 10. True. You are just multiplying the objective function by 10 (changing units from dollars to dimes).

(c) The solution to the assignment problem will also solve the assignment problem obtained by adding 4 to all benefits. True. You are just adding a constant ($4N$) to the objective function.

(d) The solution to the assignment problem will also solve the assignment problem obtained by adding 2 to $a(1, j)$ for all $j = 1, \ldots, N$. True. You are just adding 2 to the objective function. (It is as if person 1 receives 2 units of tax no matter who she matches.)

(e) Suppose that the solution to the assignment problem involves assigning Person 1 to Job 1. If $a(1, 1)$ goes up by 1, the value of the problem must increase by exactly one.

True. This change only changes the value of assignments involving matching Person 1 to Job 1. So either the value goes up by 1 (if you use the old match), or it goes up by no more than one (if you continue to match Person 1 to Job 1, but change other aspects of the match), or it does not go up at all (and may go down) (if you do not match Person 1 to Job 1).

(f) Suppose that the solution to the assignment problem involves assigning Person 1 to Job 1. If $a(1, 1)$ goes down by 1, the value of the problem must decrease by exactly one.

False. Value goes down by one if you use the same match, but it could go down by less if you change the assignment.

Iteration	1	2	3	4	5	6
1	0^*	6	2^{**}	4	8	3
2	0^*	5	2^*	4	7	3^{**}
3	0^*	5	2^*	4^{**}	7	3^*
4	0^*	5^{**}	2^*	4^*	6	3^*
5	0^*	5^*	2^*	4^*	6^{**}	3^*

The table above comes from a computation for a shortest-route problem, using the algorithm that I presented in class. The nodes in the network are connected by directed edges that have non-negative costs. It is possible that there is no direct path from Node i to Node j. In this case, $c(i, j) = \infty$.

7. (a) What is the cost of the shortest route from Node 1 to Node 5?

The cost is 6.

(b) What is the shortest route from Node 1 to Node 5?

Start at 1, go to 4, then go to 5.

(c) Is it possible to infer from the table the cost of the shortest route from Node 3 to Node 2? If so, what is the cost, what is the corresponding route, and why do you know it. If not, can you provide an upper bound for the cost? Why are you not sure whether it is actually the lowest cost.

Yes it is possible. The shortest route from 1 to 2 is to go from 1 to 3 and then to 2 (the upper bound of the cost from 1 to 2 goes down at the time that node three receives its permanent label). Consequently, the shortest route from 3 to 2 must be the direct route (otherwise, there would be a shorter way to get from 1 to 2.)

(d) In the grid below fill in as many of the costs as you can using the information in the table above. (Put the value for $c(i, j)$ in the ith row and the j th column. If you do not have enough information, write "NEI" (for "not enough information"). Notice that I filled in the diagonal elements: $c(i, i) = 0$ for all i.)

i/j	1	2	3	4	5	6
1	0	6	2	4	8	3
2	NEI	0	NEI	NEI	NEI	NEI
3	NEI	3	0	NEI	5	NEI
4	NEI	NEI	NEI	0	2	NEI
5	NEI	NEI	NEI	NEI	0	NEI
6	NEI	NEI	NEI	NEI	NEI	0