

Econ 172A, F2008: Midterm Examination II, Possible Answers

**Comments.** General information on distribution of scores. High 70, Low 12, Max 81, Mean 51, Median 52, Standard Deviation, 9.9.

Progress Report (Grading Information): I computed total points so far (sum of two midterm scores and .6 times sum of two homework scores). The total possible score for the course is 300. The total points available thus far is 174. The quintiles on “score so far” for the class are: 75, 96, 107, 119. That is, if your total score is greater than or equal to 119 you are in the top 20% of the class; greater than equal to 107, you are in the top 40% of the class; and so on.

Assigning grades for Midterm 2: We tried to be generous and sympathetic when grading the first question. The last two parts had confusing language and there were problems with the answer report associated with form B of the exam. The second and third questions were straightforward (to grade).

I hope the answers below indicate why there are deductions. Here are some comments on the first question: On (b), you can read off the value from the spreadsheet with the problem (no need to compute); on (e) even though the change is in the allowable range, the solution (to the primal) does change (this was not asked, but several people “volunteered” the wrong answer. There was no deduction, but please get the ideas straight); (g) was hard: you needed to note that the change influenced two things simultaneously so you did not have enough information to answer the question. Comments about allowable increase and allowable decrease were not relevant; (h) changed an existing constraint. If you did not state this you received little or no credit; (i) I hoped that you would recognize that this change simply increased a RHS constant; (j) this change simply scales down the production plan. Any appeal to allowable decreases is irrelevant.

1. W Oil has 5000 barrels of crude oil type 1 and 10,000 barrels of crude oil type 2 available. W sells gasoline and heating oil. These products are blends of the two crude oils. Each barrel of crude oil I has quality level 10 and each barrel of crude oil II has quality level 5. Gasoline must have an average level of at least 8 while heating oil must have an average quality level of at least 6. Gasoline sells for \$25 per barrel, and heating oil sells for \$20 per barrel. Demand for heating oil and gasoline is unlimited. I have formulated a linear programming problem that describes W's revenue-maximizing problem below.

Define  $x_{ij}$  to be the amount of oil type  $i$  used in blend  $j$ ,  $i = 1$  or  $2$ ;  $j = G$  (for gasoline) or  $O$  (for heating oil). So, in particular,  $x_{1G}$  is the amount of crude oil type 1 put in the blend for gasoline and  $x_{1G} + x_{2G}$  is the total amount of gasoline produced.

On form B, the amount of crude 2 was 12000. On form C the price of heating oil was \$15.

$$\begin{array}{rllllll}
 \max & & 25x_{1G} & + & 20x_{1O} & + & 25x_{2G} & + & 20x_{2O} & & & & & & & & & & & \\
 \text{subject to} & & x_{1G} & + & x_{1O} & & & & & & & & & & & & & & & & & & \leq & 5000 \\
 & & & & & & & & & & & & x_{2G} & + & x_{2O} & & & & & & & \leq & 10000 \\
 & & -2x_{1G} & & & & + & 3x_{2G} & & & & & & & & & & & & & & \leq & 0 \\
 & & & & -4x_{1O} & & & & & + & x_{2O} & & & & & & & & & & & \leq & 0
 \end{array}$$

and  $x \geq 0$ . Use the attached sensitivity table to answer the questions that follow. You may not have enough information to answer each part completely, but you should provide as much relevant information as possible. (For example, if you do not know the new value, but you know that the value must decrease, or must decrease by at least 100, then you should say this.) Justify your answers.

- (a) Explain the third constraint ( $-2x_{1G} + 3x_{2G} \leq 0$ ).

The constraint guarantees that the average quality of Gasoline is at least 8. To see this note that the average quality of Gasoline is

$$\frac{10x_{1G} + 5x_{2G}}{x_{1G} + x_{2G}}$$

and the constraint is just a simplified version of the restriction that this number is at least 8.

- (b) What is the solution to the problem and the associated value?

$x_{1G} = 3000$ ,  $x_{1O} = 2000$ ,  $x_{2G} = 2000$ , and  $x_{2O} = 8000$  (so you produce 5000 barrels of gasoline and 10000 barrels of heating oil. The value is \$325,000.

Form B<sup>1</sup> :  $x_{1G} = 2400$ ,  $x_{1O} = 2600$ ,  $x_{2G} = 1600$ , and  $x_{2O} = 10400$ . The value is \$360,000.

Form C: Same solution, value: \$275,000.

- (c) Write the dual of the problem.

$$\begin{array}{rllllll}
 \min & & 5000y_1 & + & 10000y_2 & & & & & & & & & & & & & & & & & & & \\
 \text{subject to} & & y_1 & & & & & - & 2y_3 & & & & & & & & & & & & & & \geq & 25 \\
 & & y_1 & & & & & & & & - & 4y_4 & & & & & & & & & & & \geq & 20 \\
 & & & & & & y_2 & + & 3y_3 & & & & & & & & & & & & & \geq & 25 \\
 & & & & & & y_2 & & & + & y_4 & & & & & & & & & & & \geq & 20
 \end{array}$$

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<sup>1</sup>Here and below the answers to the Form B questions follow from the correct information on the posted spreadsheet. The information provided with the exam was not correct. Answers that showed correct reasoning given the available data were given full credit.

and  $y \geq 0$ .

Form B: The 10,000 in the objective function becomes 12,000.

Form C: The 20s become 15s.

- (d) What is the solution to the dual and the associated value?

$y = (30, 17.5, 2.5, 2.5)$  value \$325,000 (same value as primal). Form B:  $y = (30.25, 17.5, 2.5)$  value \$500,000 (same value as primal).

Form C:  $y = (35, 10, 5, 5)$  value \$275,000 (same value as primal).

- (e) How would the value of the problem change if there were 8,000 barrels of Crude oil I available (instead of 5,000)?

You are increasing the supply by 3,000, which is less than the allowable increase. Hence values go up by 3,000 times the shadow price of crude oil I, \$30. So value goes up by \$90,000 to \$415,000.

Form B: As above.

Form C: You are increasing the supply by 3,000, which is less than the allowable increase. Value must go up by 3,000 times the shadow price of \$35, a total of \$105,000.

- (f) How would the value of the problem change if there were 2,000 barrels of Crude oil I available (instead of 5,000)?

This is a decrease of 3,000, which is outside of the allowable range. You lose \$30 times 2,500 for the first 2,500 units and at least this much for the remaining 500 units. So profit goes down by at least to \$235,000.

On B: As above except that the allowable range is 2,000.

On C: This is still a decrease of 3,000, which is outside of the allowable range. You lose \$35 times 2,500 for the first 2,500 units and at least this much for the remaining 500 units. So profit goes down by at least by \$105,000 (to \$160,000).

- (g) How would the value of the problem change if the price of Gasoline went up to \$30/barrel (from 25)?

This change influences two coefficients at once, so the sensitivity table does not help. We do know that it is feasible to continue the previous production plan, which means profit must go up by at least 5,000 times \$5 or \$25,000.

On B, C: Same.

- (h) How would the value of the problem change if the average quality of Gasoline had to be 8.5 (instead of 8)?

I don't know. The current production plan would not be feasible, so profits must go down. The shadow price does not help because this is a complicated change of the constraint (not just a change in the right-hand side).

On B, C: Same.

- (i) The government enforces average quality standards by monitoring the mixtures. It assigns ten points for each barrel of crude I in the mixture and five points for each barrel of crude II in the mixture. It then requires that the total number of points in the gasoline mixture be at least eight times the number of barrels sold (and the total number of points in the heating oil mixture be at least six times the number of barrels sold). Suppose that the government regulator gives the company 100 bonus points in the production of gasoline. That is, instead of requiring that the average quality of gasoline be at least eight, it requires that the total number of points in gasoline

plus the 100 bonus points is at least eight times the number of barrels of gasoline sold. How would the value of the problem change?

The change is equivalent to increasing the right-hand side of the average quality constraint. Since the shadow price is 2.5 and 100 is in the allowable range, this change increases profits by \$250 to \$325,250.

On B, C: same interpretation. On B, shadow price is \$2.5 and 100 is in the allowable range (on this form I asked the question about heating oil), so the change is worth \$250. On C, Shadow price is 5 and 100 is in allowable range, so the change is worth \$1000 (profits increase to \$276,000).

- (j) What happens to profits if the total amount of crude oil (both 1 and 2) decreases by 90% (so instead of having 5,000 and 10,000 barrels of crude oil 1 and 2 available, you have 500 and 1,000 barrels available)?

This just divides profits by 10 (you simply scale down production).

2. I solved a version of a linear programming problem using Excel. I attach the sensitivity report from Excel. In these reports, I replaced several values with question marks (??). Your job is to replace these question marks with the correct information. I have not given you enough information to reconstruct the problem. You should fill in the missing values using your knowledge of Excel, duality theory, and complementary slackness. You may not have sufficient information to complete the table. If you cannot determine some of the missing numbers, then say so. If you can fill in a value, then explain what permitted you to do so.

The reduced cost of  $x_1$  and  $x_2$  must be zero by complementary slackness. Similarly, complementary slackness implies that  $x_3 = 0$ . The allowable decrease for  $x_3$  is infinite because  $x_3 = 0$ . I do not know how to compute the allowable increases for  $x_2$  and the allowable decrease for  $x_1$ . The allowable increase for the coefficient of  $x_3$ , because  $x_3 = 0$  is the negative of reduced cost (4). The value of the primal is:  $60 \times 5 + 15 \times 8 + 0 \times 4 = 420$ .

Since there is slack in the first constraint, the shadow price must be zero, the allowable increase must be equal to infinity, and the allowable decrease must be equal to the slack,  $220 - 90 = 130$ . Similarly, the allowable increase for the third constraint is infinite and the allowable decrease is  $2887.5 = 3000 - 112.5$ . The RHS constraint for the fourth constraint must be -60 by complementary slackness (no slack if shadow price is positive). The shadow price for the third constraint must be 0 because there is slack. Now we can compute the shadow price of the second constraint. We know that the primal and dual have the same value. The dual value is therefore 420, which in turn must equal

$$0 \times 220 + y_2 \times 480 + 0 \times 3000 + 1 \times -60 = 480y_2 - 60,$$

so  $y_2 = 1$ . I do not know how to find the allowable increase for the second constraint.

On Form B I rearranged the rows in the constraint table and changed the RHS of the new first constraint. This lowers the allowable decrease by 1000.

On Form C, I changed the first objective coefficient from 5 to 6. This adds 60 to the objective function value. So  $y_2 = 540/480 = 1.25$ .

3. For each of the statements below indicate whether the statement is always **TRUE**, by writing “TRUE” otherwise write “FALSE.” No justification is required. The statements refer to the linear programming problem (LP) written in the form:

$$\max c \cdot x \text{ subject to } Ax \leq b, x \geq 0$$

and an associated integer programming problem (IP):

$$\max c \cdot x \text{ subject to } Ax \leq b, x \geq 0, x \text{ integer}$$

$c$  is a vector with  $n$  components,  $b$  is a vector with  $m$  components, and  $A$  is a matrix with  $m$  rows and  $n$  columns. Assume that both problems are feasible.

- (a) If the value for the (LP) is an integer, then the value of (LP) is equal to the value of (IP).  
This is false. The value of the (LP) would be an upper bound to the value of the (IP), but it is possible that the integer value came from using non-feasible (non-integer) values of the variables. (Consider  $\max 2x$  subject to  $0 \leq x \leq 1.5$ ,  $x$  integer. The value of the associated LP is 3, an integer, but the value of IP is 2.)
- (b) If a non-integer solution is feasible for (LP), then the integer solution obtained by rounding down all of the variables to the next lowest integer is feasible for (IP).  
No. This would be true if all of the entries in  $A$  were non-negative.
- (c) The feasible set for (LP) contains the feasible set for (IP).  
This is true ((IP) has more constraints).
- (d) If every  $x$  that is feasible for (LP) satisfies  $x_i \leq 10$  for  $i = 1, 2, \dots, n$ , then there are at most  $20^n$  feasible points that are feasible for (IP).  
This is true (in fact there are at most  $11^n$  feasible points: each component can take on at most one of 11 values:  $0, 1, \dots, 10$ )
- (e) If the value of the problem:

$$\min c \cdot x \text{ subject to } Ax \leq b, x \geq 0$$

is equal to -1.5, then the value of (IP) is greater than or equal to -1.

This is false. It would be true if the problem explicitly stated that the entries of  $A$ ,  $b$ , and  $c$  are integers. Here is why: The corresponding integer minimization problem cannot have a fractional value. Since the problem is feasible (this was assumed) and has more constraints than the linear programming problem, the solution to the minimization problem must be at least -1. Hence the solution to the maximization problem must be at least -1.

- (f) If  $b$ ,  $c$ , and all of the entries in  $A$  are integers, then the solution of (IP) is a solution to (LP).  
This is false. Let (LP) be  $\max x$  subject to  $2x \leq 1$ ,  $x \geq 0$ . The solution to (LP) is  $x = .5$  with value .5. The solution to (IP) is  $x = 0$ .

Form B: the questions are in a different order and there are two minor changes in wording, but the concepts are the same. Answers are: F, F, F, F, T, T. Same for C: F, F, F, F, T, T.