Econ 172A, F2008: Midterm Examination I, Possible Answers

Results 81 points possible. High: 76; Low: 13; Median: 46.

As a rough guide (no guarantees), the lowest A is about 60; the lowest B about 46; and the lowest C is about 30. These are approximations consistent with past grading norms (see course outline for more details). Your final grade will be based on your numerical scores. There is no letter grade officially associated with the midterm.

Procedures See hand out on Regrading for instructions.

Comments Here are some common mistakes.

1. Some people added non-negativity constraints while others mistakenly had a constraint describing the wrong half plane.

   Any point in the feasible set can be a solution (for some objective function). Uniqueness requires that the solution is at a corner.

   Since the feasible set is non-empty, the problem must be feasible. Since the feasible set is unbounded (if drawn correctly!), it is possible to find an objective function that makes the problem have no solution (because the value of the objective function can be made arbitrarily large).

2. Some people neglected to include non-negativity constraints in the dual. Others did not apply the complementary slackness conditions correctly. Remember you *conclude* that a constraint is binding or that a variable is zero. You never can use complementary slackness to conclude that a constraint is not binding or a variable is positive. After one solves for a solution to a subset of the dual constraints, you must check to see whether the values for the dual variables actually satisfy the remaining dual constraints. (Otherwise you do not know if these values are feasible for the dual.) Many neglected to do this.

3. Many people wrote down incorrect expressions for average quality. The constraints that described "having enough good A" neglected the need to produce A as an input for C.

Note, there are three forms. The answers below correspond to Form A. (With information about how points were allocated.) Differences between the forms

Relative to Form A, Form B is different because: (1) x_1 and x_2 are reversed and the order of some of the questions are changed; (2): The signs of the coefficients of x_2 and x_3 in the objective function are transposed; (3): the profit contributions and capacities are changed.

Relative to Form A, Form C is different because: (1) x_2's coefficients are doubled (so the values of x_2 are cut in half; (2): The coefficients of x_1 and x_4 are transposed; (3): the variable L is called P, the capacities and the average quality are changed.

1. Consider the following linear programming problem. Find x_1 and x_2 to solve:

$$\begin{array}{rrcrcr} & \max x_0 & & & & \\ \text{subject to} & 5x_1 & + & x_2 & \leq & 45 \\ & x_1 & - & x_2 & \geq & 5 \\ & -x_1 & + & 3x_2 & \leq & -5 \end{array}$$

(a) The feasible set has corners $(5, 0)$ and $(35/4, 5/4)$. It is unbounded (because x is not constrained to be nonnegative), containing points below the ray starting at $(35/4, 5/4)$ and going through $(9, 0)$ and to right of the ray that starts to $(5, 0)$ and goes through $(0, -5)$.

   i. $x_0 = x_1 + x_2$ is maximized at $(35/4, 5/4)$. The solution is unique and the value is 10.

   ii. $x_0 = x_2$ is maximized at $(35/4, 5/4)$. The solution is unique and the value is $5/4$.

   iii. $x_0 = 2x_1 + x_2$ is unbounded. Points of the form $(M, 45 - M)$ are feasible (if $M \geq 35/4$) and lead to objective function value $M + 45$.

(b) i. no (not feasible; violates the third constraint)

   ii. yes (corner, so can be unique)

   iii. yes (but not unique, since not a corner)

   iv. no (not feasible, violates the second and third constraints)

   v. yes (feasible, but interior, so cannot be unique)

(c) Answers in previous part. For (ii) and (iii) there are many possible objective functions; for (v) the objective function must be constant.

(d) No. The feasible set is not empty. Changing the objective function does not change this.

(e) Yes. See b(iii).

2. Consider the linear programming problem:

$$\begin{array}{lrcrcrcrcl} \max & 2x_1 & + & 4x_2 & - & 6x_3 & + & 5x_4 & & \\ \text{subject to} & x_1 & + & 4x_2 & + & 8x_3 & - & 2x_4 & \leq & 2 \\ & -x_1 & + & 2x_2 & + & 4x_3 & + & 3x_4 & \leq & 1 \\ & & & & & & & x & \geq & 0 \end{array}$$

(a) Write the dual of the problem.

$$\begin{array}{lrcrcr} \min & 2y_1 & + & y_2 & & \\ \text{subject to} & y_1 & - & y_2 & \geq & 2 \\ & 4y_1 & + & 2y_2 & \geq & 4 \\ & 8y_1 & + & 4y_2 & \geq & -6 \\ & -2y_1 & + & 3y_2 & \geq & 5 \\ & & & y & \geq & 0 \end{array}$$

(7 points: One point deduction for omitting non-negativity constraint; two point deduction for mistakes with the direction of constraints)

(b) Show that $x^* = (x_1^*, x_2^*, x_3^*, x_4^*) = (8, 0, 0, 3)$ is feasible for the original problem.

Plug into constraints to confirm that the two resource constraints bind and the non-negativity constraints hold. (3 points)

(c) Use complementary slackness to determine whether $(8, 0, 0, 3)$ is a solution to the original problem.

Because the first and fourth primal variables are positive, if $(8, 0, 0, 3)$ is a solution, then the first and fourth dual constraints must bind. Hence the dual solution must satisfy: $y_1 - y_2 = 2$ and $-2y_1 + 3y_2 = 5$ or $(y_1, y_2) = (11, 9)$. These satisfy the other constraints of the dual (non-negativity and the second and third resource constraints), so it is in fact a solution to the dual.

(d) If $(8, 0, 0, 3)$ is a solution to the original problem, find a solution to the dual. If $(8, 0, 0, 3)$ is not a solution to the original problem, find $x = (x_1, x_2, x_3, x_4)$, such that $x \neq x^*$; x is feasible for the original problem; and x yields a higher value of the objective function than x^* (that is, $2x_1 - 4x_2 - 6x_3 + 5x_4 > 2x_1^* - 4x_2^* - 6x_3^* + 5x_4^*$).

Fifteen points total for (c) and (d). Four points for correctly noting which dual constraints bind. Four points for correctly solving for values of dual variables. Four points for checking feasibility of these values. Three points for drawing the correct conclusion (using the appropriate information).

3. A local company can produce three products, A, B, and C. The company can sell up to 2000 units of Product A, up to 2000 units of Product B, and up to 4000 units of Product C. Each unit of Product C uses 2 units of A and 3 units of B and also requires an expenditure of \$5 (production costs). Products A and B can be produced from either Process I or Process II (or combinations of these two processes). In Process I the Company can produce two units of A and three units of B for \$6. In Process II, the company can produce one unit of A and two units of B for \$4. The unit prices for the products are \$5 for A, \$4 for B, and \$25 for C. The quality levels of each product are: A, 8; B, 7; C, 6. The average quality level of the units sold must be at least 7.

   Let x_i be the number of units of product i sold for $i = A, B, C$.

   Use the following definitions: Let x_i be the number of units of product i sold for $i = A, B, C$. Let L_j be the level Process j is operated, for $j = I$ and II. Use the variables x_A, x_B, x_C, L_I, and L_{III} to answer the problems below.

   (a) Write a constraint that guarantees that the average quality level of the units sold must be at least 7.

   $$8x_A + 7x_B + 6x_C \geq 7(x_A + x_B + x_C)$$

   or

   $$x_A - x_C \geq 0.$$

   Four points for correct answer (algebraically equivalent forms ok). Partial credit for "close" answers, for example $8x_A + 7x_B + 6x_C \geq 7$. Form C: 8 replaces 7.

   (b) Write down an expression for the profit.

   $5x_A + 4x_B + 25x_C - 6L_I - 4L_{II} - 5x_C$

   Four points total. Three points for minor errors in coefficients and two points for incorrect specification of cost. Form B: coefficients of x_A and x_B reversed.

   (c) Write down an inequality that guarantees that Processes I and II operate at high enough levels that enough of Product A is produced to sell x_A, x_B, and x_C units of Products A, B, and C respectively.

   $x_A + 2x_C \leq 2L_I + L_{II}$

   Seven points. Little partial credit (you needed to explain what you were doing).

   (d) Using the previous answers (and possibly adding additional constraints), write down a linear programming problem that the company would solve to maximize profits subject to all of the constraints given in the problem.

$$
\begin{array}{lccccccccccc}
\text{max} & 5x_A & + & 4x_B & + & 20x_C & - & 6L_I & - & 4L_{II} & & \\
\text{subject to} & -x_A & & & + & x_C & & & & & \leq & 0 \\
 & x_A & & & + & 2x_C & - & 2L_I & - & L_{II} & \leq & 0 \\
 & & & x_B & + & 3x_C & - & 3L_I & - & 2L_{II} & \leq & 0 \\
 & x_A & & & & & & & & & \leq & 3000 \\
 & & & x_B & & & & & & & \leq & 2000 \\
 & & & & & x_C & & & & & \leq & 2000 \\
 & & & & & & & & x & L & \geq & 0
\end{array}
$$

(Slight variations for different forms of the exam.) The last three resource contraints reflect the capacity constraints. The constraint above that guarantees that the processes operate at a sufficiently high level to produce enough B needed for sale and as an input to the production of C. A total of 10 points for this part, with partial credit for subsets of constraints.