Econ 172A, Fall 2008: Problem Set 3, Possible Answers

Comments:

1. Consider the linear integer programming problem:

$$\begin{array}{lrcrcr} \max & 22x_1 & + & 8x_2 & & \\ \text{subject to} & 5x_1 & + & 2x_2 & \le & 16 \\ & 2x_1 & - & x_2 & \le & 4 \\ & -x_1 & + & 2x_2 & \le & 4 \end{array}$$

   Where x_1 and x_2 are constrained to be non-negative integers.

   The feasible set for the relaxed problem (no integer constraints) consists of the polygon with corners at $(0,2),(2,0),(2,3)$, and $(8/3,4/3)$. The solution to the relaxed problem is at $(8/3,4/3)$. (Draw the graph or plug the vertices in the objective function.) The nearest integer point to $(8/3,4/3)$ is $(3,1)$. It is not feasible (violates the second constraint). The other "neighboring" points are: $(3,2)$ (not feasible); $(2,1)$ (feasible); and $(2,2)$ (feasible). $(2,2)$ yields a higher value than $(2,1)$, so it is the best guess for a solution to the integer programming problem.

   To see if this really solves the problem we need to do more work. You can identify the solution (to the integer programming problem) graphically, but I asked for branch and bound.

   If you set $x_1 = 2$ and solve the remaining problem (in relaxed form) you get $x_2 = 3$. This is a feasible integer bound with value 68. The question is: can you do better with any other choice of x_1? By the first constraint, the only feasible values for x_1 are 0, 1, and 2. (To show that $x_1 = 3$ is not feasible note that the first two constraints are inconsistent when $x_1 = 3$.) Branching on these yields the best solution to the derived linear program: $(0,2)$ and $(1,2.5)$. The values from these are less than 68, so $(2,3)$ must be the solution to the integer programming problem.

2. A machine shop makes two products. Each unit of the first product requires three hours on Machine 1 and two hours on Machine 2. Each unit of the second product requires 2 hours on Machine 1 and 3 hours on Machine 2. Machine 1 is available (at most) 8 hours each day. Machine 2 is available (at most) 7 hours each day. The profit per unit sold is 16 for the first product and 10 for the second product. The amount of each product produced per day much be an integer multiple of .25. The objective is to determine the daily mix of production quantities that will maximize profit. Formulate an integer programming problem that describes this problem. Solve the problem using the branch-and-bound technique. (You may solve the associated relaxed linear programming problems either by graphing or by using Excel.)

   Let x_i be the number of units of problem i (for $i = 1$ and 2). The machine-time constraints are straightforward (I hope) and an elegant way to express the constraint that x_i is a multiple of .25 is to say that $4x_i$ is an integer. So we have:

$$\begin{array}{lrcrcr} \max & 16x_1 & + & 10x_2 & & \\ \text{subject to} & 3x_1 & + & 2x_2 & \le & 8 \\ & 2x_1 & + & 3x_2 & \le & 7 \end{array}$$

   where x_1 and x_2 are constrained to be non-negative and $4x_i$ is an integer for $i = 1,2$, integers.

   The feasible set has corners $(0,0)$, $(0,7/3)$, $(8/3,0)$, and $(2,1)$. The solution to the relaxed problem is at $(8/3,0)$ and has value $128/3$, which is between 42.67. Since we are constrained to have the variables be multiples of .25, the value must be a multiple of .25 as well (in fact, since the coefficients in the objective function are even, the value must be a multiple of .5). This means that an upper bound for the solution of the integer problem is 42.5. $(2,1)$ yields the value 42, so there is still work to be done. The branch-and-bound technique requires that you solve the relaxed problem for each value of x_1 from 0 to 2.5 in increments of .25 (higher values for x_1 are not feasible. That is, assign a value to x_1 and then solve the problem for the best x_2 that satisfies the constraints (ignoring the "multiple of .25" constraint. So, the possible choices are

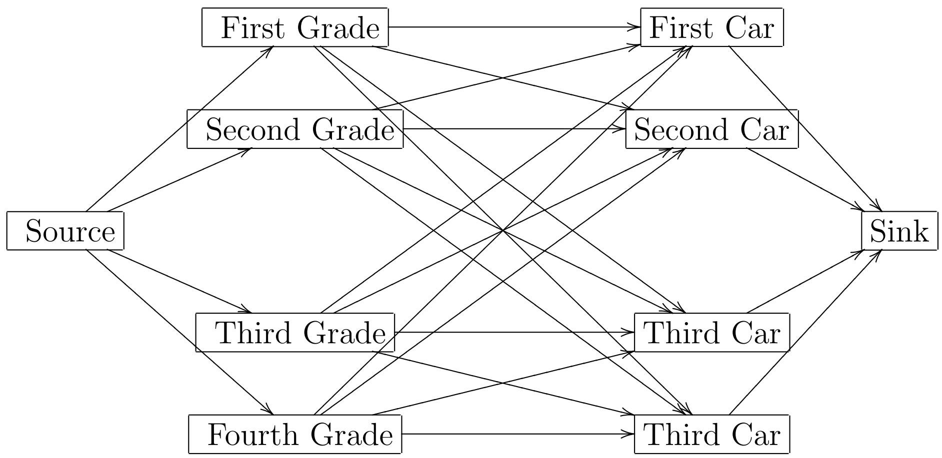
x_1	x_2	Relaxed Value
0	7/3	70/3
.25	13/6	77/3
.5	2	28
.75	11/6	91/3
1	5/3	98/3
1.25	3/2	35
1.5	4/3	112/3
1.75	7/6	119/3
2	1	42
2.25	5/8	42.25
2.5	1/4	42.5

The first column just lists possible values for x_1. The second column is the highest value for x_2 given the value of x_1 and the two constraints (since the objective function is increasing in x_2 you want the highest possible value). Notice that some of these values violate the constraint that x_2 must be a multiple of .25 – this means that these are solutions to relaxed problems and give upper bounds to the true value of the integer programming problem. The third column comes from plugging the x_1 and x_2 values into the objective function. Notice that the highest value, 42.5 is known to be the upper bound for the problem and is feasible. So the solution is $(x_1, x_2) = (2.5, .25)$ and the value is 42.5.

3. Four students from each of four grades (a total of 16 students) are eligible to go on a field trip. Four cars are available to transport the students. Two cars can carry four people each. Two cars can carry three people each. School rules require that no more than two people from each grade travel in the same car. Use a network-flow model to determine the maximum number of people than can go on the field trip. (You may assume that there are at least 16 students in each grade.)

   The point of the problem was setting it up. After you have set the problem up (maybe even before you set the problem up) it is easy to solve. For example, you can put two people from first and second grades in each of the two four-seaters. Then you can put two third graders and one fourth grader in the third car, and one third grader and 2 fourth graders in the fourth car. This carries 14 people. It is clear that you cannot do better.

   Here is the network:



The capacities on all of the edges coming from the source are four (number of students in each grade). The capacities on all of the edges coming from the second column of nodes (grade) is two (at most two students per grade in a car). The capacities on the edges comes from the third column of nodes (car) is four for two and three for the other two.

4. Five employees are available to perform four jobs. The time it takes each person to perform each job is given in the table below. Determine the assignment of employees to jobs that minimizes the total time required to perform the four jobs. (Dashed lines appear when it is impossible to schedule a person to a job. That is, Person 2 is unable to do Job 2.)

   I replaced the dashed lines by 100 (a large number) and added a column for the "waste" job to be assigned to the extra worker. This job takes no time.

	Job 1	*Job* 2	*Job* 3	*Job* 4	*Waste*
Person 1	22	18	30	18	0
Person 2	18	100	27	22	0
Person 3	26	20	28	28	0
Person 4	16	22	100	14	0
Person 5	21	100	25	28	0

   Next, I subtracted a constant from each column to create a zero in each column (leaving all numbers non-negative):

	Job 1	*Job* 2	*Job* 3	*Job* 4	*Waste*
Person 1	6	0	5	4	0
Person 2	2	82	2	8	0
Person 3	10	2	3	14	0
Person 4	0	4	75	0	0
Person 5	5	82	0	14	0

   At this point I can assign Person 4 to Job 1, Person 1 to Job 2, and Person 5 to Job 3 at zero cost, but it is not possible to cover Job 4 at zero cost too. If you cross out all of the zeros by crossing Jobs 2, 3, and 5 and Person 4, the lowest uncrossed number is 2. Subtracting this from everything and adding it back to the crossed cells once and the double-crossed cells twice yields:

	$Job\ 1$	$Job\ 2$	$Job\ 3$	$Job\ 4$	$Waste$
$Person\ 1$	4	0	5	2	0
$Person\ 2$	0	82	2	6	0
$Person\ 3$	8	2	3	12	0
$Person\ 4$	0	6	77	0	0
$Person\ 5$	3	82	0	12	0

Now I have a zero cost match: Persons 1, 2, 3, 4 and 5 get matched, respectively, to Jobs 2, 1, 5, 4, 3. The time is $18 + 18 + 14 + 25 = 75$.

5. Suppose it costs \$30,000 to purchase a new car. The annual operating cost and resale value of a used car are shown in the table below. The numbers in the "Resale Value" column indicate the amount that you can sell a car that in n years old (so a three-year old car can be sold for \$12,000). The numbers in the "Operating Cost" column indicate the amount you must pay to operate a car in its nth year of service. That is, you pay \$2,400 in the third year of service and a total of \$4,800 to maintain a three-year old car.) Assuming that you have a new car at present, determine a replacement policy that minimizes your net costs of owning and operating a car for the next six years. Solve this problem as a shortest-route problem.

   The table below describes the costs of all possible replacements. The numbers in the first row (Purchase 0) are c_{0j}, where c_{0j} is the cost of buying a car at the end of Period 0 and holding it until the end of Period j. So, for example, $c_{03} = 30000 + 900 + 1500 + 2400 - 12000 = 22800$. I wrote the costs in 100s of dollars to save typing.

Purchase	Sale					
	1	2	3	4	5	6
0	99	144	228	294	372	468
1	−−	99	144	228	294	372
2	−−	−−	99	144	228	294
3	−−	−−	−−	99	144	228
4	−−	−−	−−	−−	99	144
5	−−	−−	−−	−−	−−	99

   Now, apply the shortest route algorithm.

Iteration	Node					
	1	2	3	4	5	6
1	99**	144	228	294	372	468
2	99*	144**	228	294	372	468
3	99*	144*	228**	288	372	438
4	99*	144*	228*	288**	372	438
5	99*	144*	228*	288*	372**	432
6	99*	144*	228*	288*	372*	432**

   The first row is the "direct route" (no intermediate replacement). To get the second row we take the minimum of the first row (direct route) with the policy of taking the best route to the starred node (1) and then the direct route thereafter. Hence the value for the second row, second column is the minimum of 144 and 99+99. The value for the second row, third column is the minimum of 228 and $99 + 144$. The other second row values are, respectively, $\min\{294, 99+228\}, \min\{372, 99+294\}, \min\{468, 99+372\}$. This computation proves that the solution to the replacement problem ending in two periods is not to replace (keep the car for two years). We continue now to find the third row. The unstarred values in the third row are the minimum of the previous value plus the direct route through the newest starred node (replace after two years). Respectively, these values are: $\min\{228, 144+99\}, \min\{294, 144+144\}, \min\{372, 144+228\}, \min\{468, 144+294\}$. Notice that this computation leads to a reduction in the cost of the problem that ends after four periods. We attach two stars to the entry in Row 3 associated with Year 3 (minimum unstarred entry in row). We update the remaining three numbers in Row 3 by checking the new option (best route to the newest starred node followed by direct path). The values for the Year 4, 5, and 6 columns of Row 3 are, respectively: $\min\{288, 228+99\}, \min\{372, 228+144\}, \min\{438, 228+228\}$. And so it goes. When you are done you find that the best policy is to replace the vehicle every two years at a total cost of \$43,200.

6. The table below gives the distances between pairs of missile silos in Utah. The government is laying cables between the six silos so that any one silo can communicate with any other. What connections should be made to minimize the total cable length (subject to all towers being connected)?

From Tower	To Tower					
	1	2	3	4	5	6
1		5	14	45	32	25
2	5		2	5	22	25
3	14	2		6	26	21
4	45	5	6		13	22
5	32	22	26	13		18
6	25	25	21	22	18	

Starting with Tower 1, I connect: Tower 2, then 2 to 3, then 2 to 4, then 4 to 5, then 5 to 6. The total cost is: $5 + 2 + 5 + 13 + 18 = 43$