Econ 172A, Fall 2008: Final Examination

1. Suppose a salesperson is located at Node 0 in the graph on the next page. Find the shortest route to each of the other locations. Be sure to explain how you arrived at your answers and why your method works. [Show your work on the next page.]

Iteration	0	1	2	3	4	5	6	7	8
1	0*	23	20**	70	∞	∞	∞	∞	∞
2	0*	23**	20*	70	∞	∞	30	30	∞
3	0*	23*	20*	70	88	33	30	28**	∞
4	0*	23*	20*	33	88	33	30**	28*	48
5	0*	23*	20*	33**	88	33	30*	28*	48
6	0*	23*	20*	33*	48	33**	30*	28*	48
7	0*	23*	20*	33*	48	33*	30*	28*	48

The array gives the minimum costs for all of the routes. Working backwards we have:

(a) Shortest route to 1: direct from 0

(b) Shortest route to 2: direct from 0

(c) $0 \to 1 \to 7 \to 3$

(d) $0 \to 1 \to 7 \to 3 \to 4$

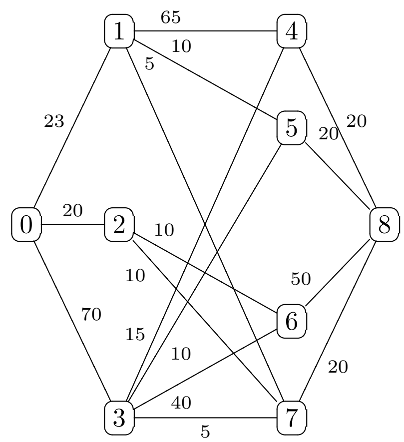
(e) $0 \to 1 \to 5$

(f) $0 \to 2 \to 6$

(g) $0 \to 1 \to 7$

(h) $0 \to 1 \to 7 \to 8$

Network for Questions 1 and 2.



2. Find the minimal spanning tree for the network in Problem 1. Your answer should identify the minimum spanning tree and its cost. You must explain the method that you use to find the solution and why the method works.

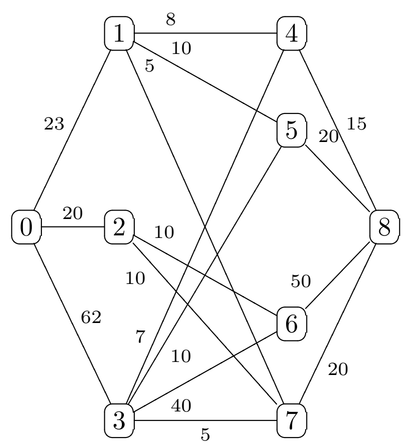
   Start at node 0; connect to 2; connect 7 to 2;connect 7 to 1; connect 3 to 7; connect 6 to 2; connect 5 to 1; connect 4 to 3; connect 8 to 4. (There are other solutions.)

   Cost:

   $20 + 10 + 5 + 5 + 10 + 10 + 15 + 20 = 95.$

3. Refer again to the network for problem 1. Assume now that the numbers on each edge are the maximum capacity that can be carried along the arc.

   Below I have reproduced the diagram, but the numbers now refer to a flow from Node 0 to Node 8.



   (a) Check that the flow is feasible. (Explicitly state everything you must check to show that the flow is feasible.)

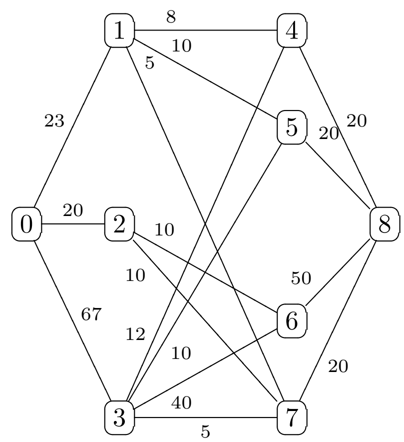
   You need to check that the flow into each node is equal to the flow out; that all flows are non-negative; and that the flow along each edge is no greater than the edge's capacity. All of these things are true.

   (b) Compute the value of the flow.

   It is 105 (sum the flow out of 0 for example).

   (c) Demonstrate that the flow is not maximal by constructing another feasible flow with a strictly greater value.

   I illustrate a better flow (obtained by using the algorithm).



(d) Is the flow you constructed in part (c) maximal? Explain.

It is. It is associated with a cut

(e) $S = \{0, 1, 3, 4\}$, $N = \{2, 5, 6, 7, 8\}$ with capacity equal to the flow (110).

(f) Find the capacity of each of the cuts below.

i. $S = \{0\}$, $N = \{1, 2, \ldots, 8\}$. Capacity: 113.

ii. $S = \{0, 1, 2 \ldots, 7\}$, $N = \{8\}$. Capacity: 110.

iii. $S = \{0, 3, 4\}$, $N = \{1, 2, 5, 6, 7, 8\}$. Capacity: 118 (43 from Node 0; 55 from Node 3; 20 from Node 4).

(g) Are any of the cuts in part (e) minimum capacity cuts for the network? Answer as completely as possible given your answers to the earlier parts of the question.

The second one is.

4. Consider the knapsack problem in which there are items $i = 1, \dots, 6$. Item i has weight w_i and value v_i.

	1	2	3	4	5	6
Weight	1	4	10	13	16	20
Value	1	6	16	26	40	31

The capacity of the knapsack is 53. You want to maximize the total value you put is the knapsack subject to the constraint that you the total weight carried must be no greater than 53. You can carry at most one unit of each item.

(a) Solve the problem assuming that you can carry fractional quantities.

Here you fill then knapsack in order of value to weight ratio. You pack items 5, 4, and 3 (in that order). This uses 39 out of 53 pounds. You then pack 14/20 of item 6. Value: $40 + 26 + 16 + 21.7 = 103.7$

(b) Now suppose that the items are not divisible: You can either carry an item or not. (This is the standard integer-constrained knapsack problem.) Use your answer to part (a) to obtain an integer upper bound to the value of the problem.

103 (round down)

(c) Perform one branching step. Give a finite lower bound to the value of the integer-constrained knapsack problem.

I choose to branch on item 3 (this is a bit dishonest – it leads to the solution quickly). If I do not take the item, then I can take everything but else but item 1 (that is, the solution to the relaxed problem is integer). The value is 103, the upper bound. [If I do take the item, then the solution to the relaxed problem is the solution from part (a).]

(d) Find the solution to the integer-constrained knapsack problem. You may use any method, but you must explain completely why your answer solves the problem.

Answered in the previous part.

5. Consider the linear programming problem:

$$\begin{array}{lrcrcrcr}
\max & 4x_1 & + & 16x_2 & + & 15x_3 & & \\
\text{subject to} & x_1 & + & x_2 & + & x_3 & \leq & 10 \\
 & x_1 & + & 6x_2 & + & 2x_3 & \leq & 26 \\
 & & & & & x_3 & \leq & 1
\end{array}$$

$$x_1, x_2, x_3 \geq 0.$$

(a) Write the dual.

$$\begin{array}{lrcrcrcr}
\min & 10y_1 & + & 26y_2 & + & y_3 & & \\
\text{subject to} & y_1 & + & y_2 & & & \geq & 4 \\
 & y_1 & + & 6y_2 & & & \geq & 16 \\
 & y_1 & + & 2y_2 & + & y_3 & \geq & 15
\end{array}$$

$$y_1, y_2, y_3 \geq 0.$$

(b) Show that $(y_1, y_2, y_3) = (1.6, 2.4, 8.6)$ is a solution to the dual by finding the solution to the given problem and invoking the relevant results from the class.

Complementary slackness says to look for something that satisfies all primal constaints as equations. If we do this, we get $(x_1, x_2, x_3) = (6, 3, 1)$. This is feasible for the primal so since y is feasible for dual, they solve their respective problems. Answer the following questions as completely as possible. You may not have enough information to answer each part completely, but you should provide as much relevant information as possible.

(c) How does the value of the problem change if the coefficient of x_3 in the objective function is increased to 20?

Profits go up by at least $(20 - 15) \times x_3 = 5$. They could go up by more (not enough information to know). [Actually, y_3 acts like a slack variable and the change only leads to a new dual solution of $y = (2.4, 1.6, 13.6)$ and the value goes up by exactly 5.]

(d) How does the value of the problem change if the constant on the right-hand side of the third constraint fell from 1 to .9

Profits fall by at least .1 (the amount of fall of constraint) times $y_3 = 8.6$ (corresponding shadow price). So they fall by at least .86. (It is not hard to compute the new solution, $x = (6.08, 3.02, .9)$. The value goes down by exactly .86.)

(e) How does the value of the problem change if I added the constraint $x_1 + x_2 + 2x_3 \leq 13$ to the primal?

It does not change at all (adding a constraint cannot raise value and previous solution is still feasible).

6. For each of the statements below indicate whether the statement is always **TRUE**, by writing "TRUE" otherwise write "FALSE." No justification is required.

   The next three parts refer to a network in which there are N nodes and in which $c(i,j)$ is the cost of going from node i to node j and all pairs of nodes are connected. Assume that $\infty > c(i,j) \geq 0$ and that the costs are distinct ($c(i,j) = c(i',j')$ if and only if $i = i'$ and $j = j'$).

   (a) Every spanning tree has exactly $N - 1$ edges.

   This is true. The first edge covers two nodes, the remaining edges cover exactly one new node.

   (b) The average cost of an edge in the minimum spanning tree is less than or equal to the Nth lowest cost of an edge in the network.

   This is false. Suppose that there are five nodes. Four of them can be connected at zero cost no matter which edge you use ($c(i,j) = 0$ provided that $i, j < 5$). There are 6 such edges, so the Nth lowest cost is 0. On the other hand, the minimal spanning tree must connect one of these nodes to the fifth node. The remaining cost could be as large as I want.

   (c) There exists a pair of nodes in the network i and j with the property that the shortest route from i to j is the direct route.

   This is true as long as the costs are all non-negative.

   The next two parts refer to a network in which there is a source s and a sink n and in which $c(i,j) \geq 0$ is the capacity of the edge going from Node i to Node j. Denote a flow by (x_{ij}), where, for each pair of Nodes i and j, x_{ij} is the amount that flows from Node i to Node j.

   (d) The value of the minimal flow is equal to the maximum cut capacity.

   This is false. (Min and max are reversed.)

   (e) Let (S, N) be a cut in which $s \in S$ and $n \in N$. If there exists a flow (x_{ij}) with the property that $x_{ij} = c_{ij}$ whenever $i \in S$ and $j \in N$, then (x_{ij}) is a maximal flow.

   This is true because the flow must be equal to the capacity of a cut (and any feasible cut capacity is an upper bound to the flow).

7. In each part indicate which of the choices are correct. (For each part, there may be 0, 1, 2, 3, or 4 correct choices.)

   (a) The inequalities below describe two-dimensional sets. Which of these sets can be described by linear inequalities?

      i. $x + y^2 \leq 4$.
      ii. $x + y \leq 4$.
      iii. $|x + y| \leq 4$.
      iv. $\frac{x}{x+y+2} \leq .4, x \geq 0, y \geq 0$.

   The correct answers are (b), (c), and (d). You can see this by graphing the inequalities. In the first case the set does not have straight line segments as boundaries. (b) is obvious. The appropriate system of linear inequalities are $x + y \leq 4$ and $x + y \geq -4$ for (c) and $-.6x + .4y + .8 \geq 0$, $x \geq 0$, $y \geq 0$ for (d).

   (b) Which of the statements below are true statements about following linear programming problem (P) or its dual (D):

$$\begin{array}{lrcrcrcrcr} \max & 15x_1 & + & 6x_2 & + & 9x_3 & + & 2x_4 & & \\ \text{subject to} & 2x_1 & + & x_2 & + & 5x_3 & + & 6x_4 & \leq & 20 \\ & 3x_1 & + & x_2 & + & 3x_3 & + & 3x_4 & \leq & 24 \\ & 7x_1 & & & & & + & x_4 & \leq & 70 \\ & & & & & & & x & \geq & 0 \end{array}$$

      i. $x^* = (4, 12, 0, 0)$ is a solution to (P).
      ii. $x^* = (10, 6, -4, 0)$ is a solution to (P).
      iii. $y^* = (3, 3, 0)$ is a solution to (D).
      iv. If the right-hand side of the third constraint increased from 70 to 85, the solution to (P) would not change.

   You can check that the first is feasible for (P), the third is feasible for (D), and they yield the same value. Further, when the RHS of the third constraint changes to 85, both of these remain feasible and yield the same value, so they must both continue to be solutions. Answer: everything but (b) (which is not feasible).

   (c) Consider the linear programming problem:

$$\max c \cdot x \text{ subject to } Ax \leq b, x \geq 0.$$

   Assume that all entries in A, b, and c are whole numbers and that the problem has a solution x^*. By a whole number solution, I mean a solution x^{**} with the property that every component of x^{**} is a whole number.

      i. The problem must have a whole number solution.
      ii. The problem must have a whole number solution provided that all of the entries in A, b, and c are equal to -1, 0, or 1.

iii. The problem must have a whole number solution provided that the dual of the problem has a whole number solution.

iv. If the problem has a whole number solution, then the value of the problem is a whole number.

(d) is the only correct choice. It is true because you get the value by multiplying the whole number solution by whole numbers and adding.

A counterexample to (a) and (c) is: $\max 2x$ subject to $2x \leq 1$, $x \geq 0$. A counterexample to (b) is: $\max x_1$ subject to $x_1 + x_2 \leq 1$, $x_1 - x_2 \leq 0$, $x, y \geq 0$.