Econ 172A, Fall 2008: Final Examination Version A

Instructions.

1. The examination has seven questions. Answer them all.

2. If you do not know how to interpret a question, then ask me.

3. Questions 1-5 require you to provide numerical answers. You must explain how you arrive at the numerical answer. If you properly use an algorithm introduced in the class that is appropriate for the problem, it is sufficient to say: "I used the algorithm introduced in class." (You still need to show the work necessary to demonstrate that you correctly carried out the steps in the algorithm.) If you use an alternate method, a detailed justification of why your method works is necessary.

4. To be clear: It is not sufficient to write down a correct numerical answer to receive credit. You must explain how you arrived at your answer and why it is appropriate.

5. No justification is needed for Questions 6 and 7. Please enter your answers to these questions in the grid following Question 7.

6. The table below indicates how points will be allocated on the exam.

7. Work alone. You may not use notes, books, calculators, or any other electronic devices.

8. You have three hours.

9. If you sign the Buckley waiver attached to the examination, then I will place your graded exam in a public place (second floor alcove in Economics Department Building). Otherwise, they will be available for pick up in the departmental office at the start of the third week of classes Winter Quarter 2009.

|	Score	Possible
I		21
II		10
III		27
IV		20
V		20
VI		10
VII		12
Exam Total		120
Course Total		300
Grade in Course		

1. Suppose a salesperson is located at Node 0 in the graph on the next page. Find the shortest route to each of the other locations. Be sure to explain how you arrived at your answers and why your method works. [Show your work on the next page.]

Network for Questions 1, 2, and 3.
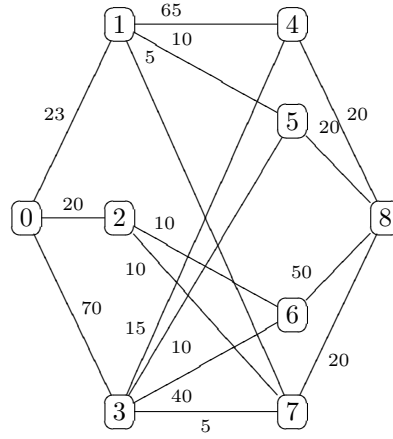
2. Find the minimal spanning tree for the network in Problem 1. Your answer should identify the minimum spanning tree and its cost. You must explain the method that you use to find the solution and why the method works.

3. Refer again to the network for problem 1. Assume now that the numbers on each edge are the maximum capacity that can be carried along the arc.

   Below I have reproduced the diagram, but the numbers now refer to a flow from Node 0 to Node 8.
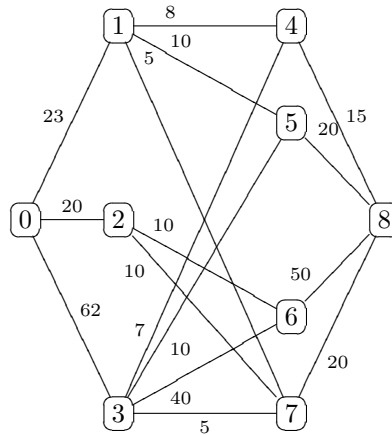
   (a) Check that the flow is feasible. (Explicitly state everything you must check to show that the flow is feasible.)

   (b) Compute the value of the flow.

   (c) Demonstrate that the flow is not maximal by constructing another feasible flow with a strictly greater value. Write the new flow in the network drawing on the next page (it is the same as the original network with capacities omitted).
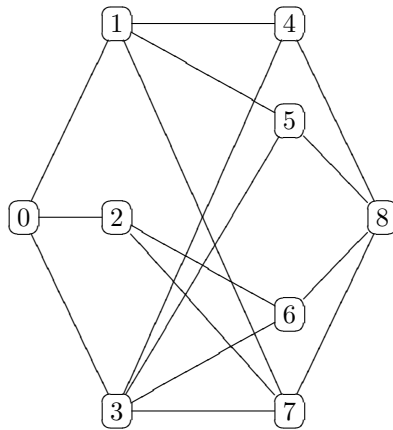
(d) Is the flow you constructed in part (c) maximal? Explain.

(e) Find the capacity of each of the cuts below.
   i. $S = \{0\}$, $N = \{1, 2, \ldots, 8\}$.

   ii. $S = \{0, 1, 2 \ldots, 7\}$, $N = \{8\}$.

   iii. $S = \{0, 3, 4\}$, $N = \{1, 2, 5, 6, 7, 8\}$.

(f) Are any of the cuts in part (e) minimum capacity cuts for the network? Answer as completely as possible given your answers to the earlier parts of the question.

4. Consider the knapsack problem in which there are items $i = 1, \ldots, 6$. Item i has weight w_i and value v_i.

	1	2	3	4	5	6
Weight	1	4	10	13	16	20
Value	1	6	16	26	40	31

The capacity of the knapsack is 53. You want to maximize the total value you put in the knapsack subject to the constraint that you the total weight carried must be no greater than 53. You can carry at most one unit of each item.

(a) Solve the problem assuming that you can carry fractional quantities.

(b) Now suppose that the items are not divisible: You can either carry an item or not. (This is the standard integer-constrained knapsack problem.) Use your answer to part (a) to obtain an integer upper bound to the value of the problem.

(c) Perform one branching step. Give a finite lower bound to the value of the integer-constrained knapsack problem.

(d) Find the solution to the integer-constrained knapsack problem. You may use any method, but you must explain completely why your answer solves the problem.

5. Consider the linear programming problem:

$$
\begin{array}{llrcrcrcl}
\max & 4x_1 & + & 16x_2 & + & 15x_3 & & \\
\text{subject to} & x_1 & + & x_2 & + & x_3 & \leq & 10 \\
& x_1 & + & 6x_2 & + & 2x_3 & \leq & 26 \\
& & & & & x_3 & \leq & 1
\end{array}
$$

$$x_1, x_2, x_3 \geq 0.$$

(a) Write the dual.

(b) Show that $(y_1, y_2, y_3) = (1.6, 2.4, 8.6)$ is a solution to the dual by finding the solution to the given problem and invoking the relevant results from the class.

Answer the following questions as completely as possible. You may not have enough information to answer each part completely, but you should provide as much relevant information as possible.

(c) How does the value of the problem change if the coefficient of x_3 in the objective function is increased to 20?

(d) How does the value of the problem change if the constant on the right-hand side of the third constraint fell from 1 to .9?

(e) How does the value of the problem change if I added the constraint $x_1 + x_2 + 2x_3 \leq 13$ to the primal?

6. For each of the statements below indicate whether the statement is always **TRUE**, by writing circling "TRUE" in the grid at the bottom of the page; otherwise circle "FALSE." No justification is required. Please mark your answers in the grid following Question 7.

The next three parts refer to a network in which there are N nodes and in which $c(i, j)$ is the cost of going from node i to node j and all pairs of nodes are connected. Assume that $\infty > c(i, j) \geq 0$ and that the costs are distinct ($c(i, j) = c(i', j')$ if and only if $i = i'$ and $j = j'$).

(a) Every spanning tree has exactly $N - 1$ edges.

(b) The average cost of an edge in the minimum spanning tree is less than or equal to the Nth lowest cost of an edge in the network.

(c) There exists a pair of nodes in the network i and j with the property that the shortest route from i to j is the direct route.

The next two parts refer to a network in which there is a source s and a sink n and in which $c(i, j) \geq 0$ is the capacity of the edge going from Node i to Node j. Denote a flow by (x_{ij}), where, for each pair of Nodes i and j, x_{ij} is the amount that flows from Node i to Node j.

(d) The value of the minimal flow is equal to the maximum cut capacity.

(e) Let (S, N) be a cut in which $s \in S$ and $n \in N$. If there exists a flow (x_{ij}) with the property that $x_{ij} = c_{ij}$ whenever $i \in S$ and $j \in N$, then (x_{ij}) is a maximal flow.

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7. In each part indicate which of the choices are correct. (For each part, there may be 0, 1, 2, 3, or 4 correct choices.) Please mark your answers in the grid following Question 7 (circle all of the correct choices).

(a) The inequalities below describe two-dimensional sets. Which of these sets can be described by linear inequalities?

    i. $x + y^2 \leq 4$.

    ii. $x + y \leq 4$.

    iii. $|x + y| \leq 4$.

    iv. $\frac{x}{x+y+2} \leq .4, x \geq 0, y \geq 0$.

(b) Which of the statements below are true statements about following linear programming problem (P) or its dual (D):

$$
\begin{array}{rrrrrrrrrrr}
\text{max} & 15x_1 & + & 6x_2 & + & 9x_3 & + & 2x_4 & & & \\
\text{subject to} & 2x_1 & + & x_2 & + & 5x_3 & + & 6x_4 & \leq & 20 \\
& 3x_1 & + & x_2 & + & 3x_3 & + & 3x_4 & \leq & 24 \\
& 7x_1 & & & & & + & x_4 & \leq & 70 \\
& & & & & & & x & \geq & 0
\end{array}
$$

    i. $x^* = (4, 12, 0, 0)$ is a solution to (P).

    ii. $x^* = (10, 6, -4, 0)$ is a solution to (P).

    iii. $y^* = (3, 3, 0)$ is a solution to (D).

    iv. If the right-hand side of the third constraint increased from 70 to 85, the solution to (P) would not change.

(c) Consider the linear programming problem:

$$\max c \cdot x \text{ subject to } Ax \leq b, x \geq 0.$$

Assume that all entries in A, b, and c are whole numbers and that the problem has a solution x^*. By a whole number solution, I mean a solution x^{**} with the property that every component of x^{**} is a whole number.

    i. The problem must have a whole number solution.

    ii. The problem must have a whole number solution provided that all of the entries in A, b, and c are equal to -1, 0, or 1.

    iii. The problem must have a whole number solution provided that the dual of the problem has a whole number solution.

    iv. If the problem has a whole number solution, then the value of the problem is a whole number.

Answer Grid for Question 6
Circle "TRUE" if the statement is always true or "FALSE" otherwise for each part.

PART		
a	TRUE	FALSE
b	TRUE	FALSE
c	TRUE	FALSE
d	TRUE	FALSE
e	TRUE	FALSE

Answer Grid for Question 7
Circle the correct statements.
Remember there may be 0, 1, 2, 3, or 4 correct statements for each part.

PART		PART		PART	
a	i	b	i	c	i
a	ii	b	ii	c	ii
a	iii	b	iii	c	iii
a	iv	b	iv	c	iv