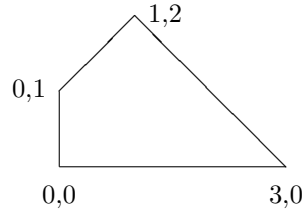


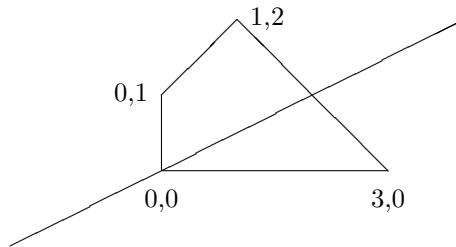
**Econ 172A, Fall 2007: Problem Set 1**

1. (a) Here is a picture. (The feasible set is the figure and its interior.)



- (b) i.  $x^* = (3, 0)$ , value = 3.  
 ii.  $x^* =$  segment connecting  $(3, 0)$  to  $(1, 2)$ , value = 3.  
 iii.  $x^* = (3, 0)$ , value = 3.  
 iv.  $x^* = (1, 2)$ , value = 3.

In these answers,  $x^*$  is the name that I give to the solution. What follows is the picture with a line in which  $x_1 - 2x_2$  is constant. Shifting the line parallel, down, and to the right increases the value, hence tells you that the solution to (iii) is  $(3, 0)$ . Shifting the line parallel, up, and to the left decreases the value and (since  $\max x_0$  is the same as  $\min -x_0$ ) provides an answer to (iv). The other two parts are similar.



- (c) The corners of the feasible set (apparent from the picture) are:  $(0, 0)$ ,  $(1, 0)$ ,  $(3, 0)$ , and  $(1, 2)$ . From (b), (i) gives an objective function with unique solution at  $(3, 0)$  and (iv) gives an objective function with unique solution at  $(1, 2)$ . For  $(0, 0)$  one possibility is  $x_0 = -x_1 - x_2$ ; for  $(1, 0)$ , one possibility is  $x_0 = -10x_1 + x_2$ . (The idea is to play around with slopes of objective functions that make different corners solutions. There are many possible solutions.)
- (d) The Excel spreadsheet contains the template for the problem. The answers are the same as the graphical answers except that Excel does not indicate multiple solutions (the particular solution the Excel finds for you will depend on how you entered the data).
- (e) There is no need to do additional computations. This change doesn't change the solution. It multiplies values by 5. (All you are doing is changing units.)

- (f) This change does absolutely nothing. The answers and values stay the same. Perhaps the quickest way to see this is to graph the “new” feasible set: It is exactly the same as the old feasible set.
- (g) What happens here is that the units of  $x_2$  only are changed. The problem would be exactly the same if I created a new variable, called it  $y_2$ , set  $y_2 = 5x_2$ , and replaced  $x_2$  by  $y_2$  everywhere in the problem. Hence the values don’t change, the  $x_1$  part of the solution doesn’t change; the  $x_2$  part is multiplied by .2. (So, for example, the solution to part b, (iv) is (1, .4).)
- (h) This change really changes the problem. For the new (f), you change the direction of the second constraint. The new solutions are
- $x^* = (1, 2)$  value = 1.
  - $x^* =$  segment between (0, 3) and (1, 2), value = 3.
  - $x^* = (0, 1)$ , value = -2.
  - $x^* = (0, 3)$ , value = 6.

For the new (g), you change the constraint set. The new feasible set is unbounded: it is bounded by (0, 0), (0, 3), the line  $x_2 = 0$ , and the line  $x_1 - 5x_2 = 3$ . Parts (a) and (c) are unbounded and part (b) is the entire ray:  $x_1 - 5x_2 = 3, x_1 \geq 0$  (value 3), and ; part (d) the solution is (0, 0, value 0. (You can tell this by graphing or using Excel.)

2. I found it useful to define three kinds of variable:  $x_S, x_C$ , and  $x_O$  are the number of acres used for soybeans, corn, and oats, respectively.  $y_H$  and  $y_C$  are the number of hens and cows.  $l_W$  and  $l_S$  are the number of unused hours of labor in winter and summer. Of course, you can denote all the variables by  $x_i$ . Also, you do not need to have separate variables for the surplus labor. I just found that using these variables makes the problem and the dual more transparent. With these definitions, the problem becomes:

	max	1000 $y_C$	+	5 $y_H$	+	500 $x_S$	+	750 $x_C$	+	350 $x_O$	+	5 $l_W$	+	6 $l_S$		
subject to		1.5 $y_C$				$x_S$	+	$x_C$	+	$x_O$					≤	125
		1200 $y_C$	+	9 $y_H$											≤	40000
		100 $y_C$	+	.6 $y_H$	+	20 $x_S$	+	35 $x_C$	+	10 $x_O$	+	$l_W$			≤	3500
		50 $y_C$	+	.3 $y_H$	+	50 $x_S$	+	75 $x_C$	+	40 $x_O$			+	$l_S$	≤	4000
		$y_C$													≤	3000
				$y_H$											≤	32
		$y_C$		$y_H$		$x_S$		$x_C$		$x_O$		$l_W$		$l_S$	≥	0

The constraints are, in order, land, investment, winter labor, summer labor, barn capacity, chicken house capacity, and nonnegativity. I think that the only possible confusion is the way that I introduced the variables  $l_W$  and  $l_S$ . These are added onto the left-hand sides of the labor constraints and also appear in the objective function. Notice that even though I wrote the labor constraints as inequalities, the constraints must bind when we solve the problem. The Foster’s won’t “throw away” labor that they could “sell” for at least \$5 per hour. Two other things to note. In my formulation I assumed that hens require .3 hours of labor in the winter. You could interpret the problem as stating that hens require .6 + .3 = .9 hours of labor in winter (that is, .3 *additional* hours. This change influences the answer to the problem, but it is a reasonable interpretation of the problem description. Finally, I have assumed that there is no value to having left over investment money. Alternatively, you might assume that any money not invested should be included in the objective function.

$$\begin{array}{rcccccccc}
\min & 125z_1 & + & 4000z_2 & + & 3500z_3 & + & 4000z_4 & + & 3000z_5 & + & 32z_6 & & \\
\text{subject to} & 1.5z_1 & + & 1200z_2 & + & 100z_3 & + & 50z_4 & + & z_5 & & & \geq & 1000 \\
& & & 9z_2 & + & .6z_3 & + & .3z_4 & & & + & z_6 & \geq & 5 \\
& z_1 & & & + & 20z_3 & + & 50z_4 & & & & & \geq & 500 \\
& z_1 & & & + & 35z_3 & + & 75z_4 & & & & & \geq & 750 \\
& z_1 & & & + & 10z_3 & + & 40z_4 & & & & & \geq & 350 \\
& & & & & z_3 & & & & & & & \geq & 5 \\
& & & & & & & z_4 & & & & & \geq & 6 \\
& z_1 & & z_2 & & z_3 & & z_4 & & z_5 & & z_6 & \geq & 0
\end{array}$$

- (b) It is convenient to treat this as a production problem. The dual variables are the prices of inputs. For example,  $z_1$  the dual variable associated with the first constraint is the value of an additional acre of land to the Foster family farm. The constraints of the dual guarantee that it is at least as profitable to sell the land at the dual prices than to operate one of the many enterprises (raising cows, planting soy) that are possible.
- (c) Solution of Primal (via Excel):  $y_C = 23.75$ ,  $x_S = 56.25$ , and profit 51875. The shadow price (dual variable) associated with winter labor is equal to \$6.25 and the price of summer labor is \$7.50. All other dual variables are zero. The value is 51875.
- (d) You should check five kinds of thing. First, the value of the primal and dual are the same. Second, if a primal variable is positive, then the associated dual constraint must bind. Since there are two positive primal variables ( $y_C$  and  $x_S$ ), check that the first and third constraints in the dual bind. Third, if a primal constraint is not binding, then the associated dual variable must be zero. The first, second, fifth, and sixth primal constraints don't bind (slack is positive). As should be the case, the associated dual variables are zero. Fourth, if a dual variable is positive, then the associated primal constraint binds. The third and fourth dual variables are positive and, indeed, these constraints bind in the primal. Finally, if a dual constraint is not binding (like all but the first and third), the associated primal variable must be zero. And they are.
3. For convenience, number the days: Monday, day 1; Tuesday, day 2; ... Sunday, day 7. Number the shifts: Night, shift 1; Day, shift 2; and Late shift 3.

Introduce the variables:

$x_{ij}$  = the number of workers starting their four consecutive work days on day  $i$  ( $i = 1, \dots, 7$ )

and shift  $j$  ( $j = 1, \dots, 3$ )

These variables should be integers. Here is where the linear programming formulation is a but nutty.

Now write the constraints: let  $D_{ij}$  be the (known) number of workers required on day  $i$  ( $i=1, \dots, 7$ ) and shift period  $j$ , so that, for example,  $D_{53} = 11$ . The constraints are:

$x_{ij} \geq 0$  for all  $i$  and  $j$ .

Monday:  $x_{1j} + x_{7j} + x_{6j} + x_{5j} \geq D_{1j}, j = 1, 2, 3$

General day  $i$ :  $x_{ij} + x_{(i-1)j} + x_{(i-2)j} + x_{(i-3)j} \geq D_{ij}, j = 1, 2, 3$ , where  $i - k$  is interpreted modulo 7 (is that if  $i = 2$  the expression is:  $x_{2j} + x_{1j} + x_{7j} + x_{6j} \geq D_{1j} j = 1, 2, 3$ ). The objective is to min  $\sum_{i=0}^7 \sum_{j=1}^3 x_{ij}$ .