Econ 172A, Fall 2007: Midterm Solutions and Grading Notes

What follows are the approximate rules used to assign points.

1. 21 points total
   (a) 4 points: 1 point for writing a dual with the correct number of variables and constraints; 1 point for objective function; 1 point for constraints; 1 point for nonnegativity constraints.
   (b) 3 points: 2 points for checking first three constraints and one for nonnegativity (if they ignored nonnegativity in the first part, do not make second deduction).
   (c) 9 points: 3 points for drawing correct inference from non binding constraints (you can give one per constraint); 3 points for drawing the correct inference from positive variable (one point per variable); 3 points for writing down the correct system to solve for dual guess (if they show their reasoning, full credit here for the system consistent with their CS deductions – that is if they make a mistake selecting which constraints bind, then do not punish them again); one point for correct answer (again, no double punishment if they reason correctly following an earlier slip). It is possible that you'll want to give some partial credit for students who write something sensible, but fail to get points according to the rules above. Talk to me if you are uncertain.
   (d) 5 points: I want the students to check the remaining constraints and draw the right conclusion. If they do this, I'd give three points for doing the checking and 2 points for drawing the right conclusion. Other answers are possible. For example, if a student exhibits a feasible x that gives value greater than 3 ($(0, 4, 0)$ is an easy to spot example), then they deserve full credit.
2. They need to provide a definition of variables, the correct objective function, and the correct constraints. I would give 3 points for the variables, 3 points for the objective function, 4 points for the constraints (notice adding non-negativity is ok), and 6 points for the second part. Partial credit for the second part is probably possible (check with me if you are not sure about how to allocate it). There is no need to simplify and it is ok to have a constant in the expression. On this question, I am happy if students come up with alternative explanations, but
3. straightforward
4. straightforward (as in the previous problem)
5. 34 points: 4, 4 (1 point for the reason), 4 (1 point for the reason), 4 (1 point for reason), 6 (2 points for recognizing this is about a particular shadow price, 2 points for checking allowable range, 2 points for multiplication – getting the right answer), 6 (4 points for figuring out the cost of the new product and 2 points for knowing what to compare it to), 6 (same as part f)

There are two forms. They differ in small ways. In the first question x_1 and x_2 are interchanged. The second part of the second question is slightly different. One of the corners in question 3 is different (leading to a change in some of the details of the subsequent questions). The order of the parts in Question 4 is different. In Question 5, I changed the price of cheeseburgers, which has an influence on some of the answers.

1. Consider the linear programming problem:

   Find x_1, x_2 and x_3 to solve **P**:

$$\begin{array}{lrcrcrcl} \max & x_1 & + & x_2 & + & 4x_3 & & \\ \text{subject to} & x_1 & + & 2x_2 & + & x_3 & \leq & 4 \\ & 2x_1 & - & x_2 & + & x_3 & \leq & 4 \\ & x_1 & + & x_2 & & & \leq & 3 \\ & & & x & \geq & 0 & & \end{array}$$

   You must provide justifications for your answers to the questions below. In particular, say what you need to do to check for feasibility and the basis for your inferences in part (c).

   (a) Write the dual of the problem **P**.

   Find y_1, y_2 and y_3 to solve **D**:

$$\begin{array}{lrcrcrcl} \min & 4y_1 & + & 4y_2 & + & 3y_3 & & \\ \text{subject to} & y_1 & + & 2y_2 & + & y_3 & \geq & 1 \\ & 2y_1 & - & y_2 & + & y_3 & \geq & 1 \\ & y_1 & + & y_2 & & & \geq & 4 \\ & & & y & \geq & 0 & & \end{array}$$

   (b) Verify that $(x_1, x_2, x_3) = (2, 1, 0)$ is feasible for **P**.

   Plainly $x \geq 0$. The first and third constraints are binding. The second one holds (with slack equal to 1).

   (c) Assuming that $(2, 1, 0)$ is a solution to **P**, use Complementary Slackness to determine a candidate solution to the dual.

   Since x_1 and x_2 are positive, the first two dual constraints must bind. Since the second primal constraint is not binding, $y_2 = 0$. Hence the solution to the dual must satisfy: $y_1 + y_3 = 1$ and $2y_1 + y_3 = 1$ so $y_1 = 0$ and $y_3 = 1$ (and $y_2 = 0$) is the candidate solution to the dual.

   (d) Is $(2, 1, 0)$ a solution to **P**? Explain.

   No. While these values are non-negative, they fail to satisfy the third constraint of the dual. Since the "guess" for the dual is not feasible, the $(2, 1, 0)$ cannot solve the primal.

2. Convex Pizza is a producer of frozen pizza products. The company makes a net income of \$1.00 for each regular pizza and \$1.50 for each deluxe pizza produced. The firm currently has 150 pounds of dough mix and 50 pounds of topping mix. Each regular pizza uses 1 pound of dough mix and 4 ounces (16 ounces = 1 pound) of topping mix. Each deluxe pizza uses 1 pound of dough mix and 8 ounces of topping mix. Based on the past demand per week, Convex can sell at least 50 regular pizzas and at least 25 deluxe pizzas. The problem is to determine the number of regular and deluxe pizzas the company should make to maximize net income. Formulate this problem as an LP problem. Your formulation should include a definition of the variables (in words).

   Let x_1 and x_2 be the number of regular and deluxe pizzas produced, then the LP formulation is:

$$\begin{array}{lrcrcr} \text{max} & x_1 & + & 1.5x_2 & & \\ \text{subject to} & x_1 & + & x_2 & \leq & 150 \\ & .25x_1 & + & .5x_2 & \leq & 50 \\ & x_1 & & & \geq & 50 \\ & & & x_2 & \geq & 25 \end{array}$$

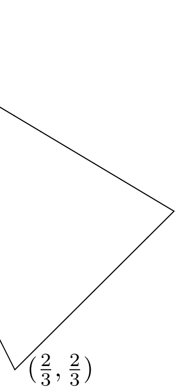
   The first constraint describes the constraint about dough mixing and the second constraint describes the resource constraint about topping. The last two constraints interpret the problem as stating that you must produce a minimum of so many regular and deluxe pizzas.

   If unused topping mix is worth \$.5 per pound, then the objective function becomes:

$$x_1 + 1.5x_2 - .5(.25x_1 + .5x_2) = .875x_1 + 1.25x_2$$

   On the other form, I asked what happens if unused dough mix was worth \$.75 per pound. This makes the objective function:

$$.25x_1 + .75x_2.$$



3. The following questions relate to the triangle above (and its interior), which I call S. You should think of S as the feasible set of a linear programming problem.

   Remember, any number of choices (from zero to four) can be correct. As described on the answer sheet, you receive credit for each correct choice you select and for each incorrect choice you do not select.

   (a) The triangle above (and its interior) is described by which of the following sets of linear inequalities.

   i. $$\begin{array}{rcrcr} 3x_1 & + & 5x_2 & \leq & 24 \\ x_1 & - & x_2 & \leq & 0 \\ 2x_1 & + & x_2 & \geq & 2 \\ & & x_2 & \geq & 0 \end{array}$$

   ii. $$\begin{array}{rcrcr} 3x_1 & + & 5x_2 & \leq & 24 \\ x_1 & - & x_2 & \geq & 0 \\ 2x_1 & + & x_2 & \geq & 2 \end{array}$$

   iii. $$\begin{array}{rcrcr} 3x_1 & + & 5x_2 & \leq & 24 \\ x_1 & - & x_2 & \geq & 0 \\ 2x_1 & + & x_2 & \geq & 2 \\ x_1 & & & \geq & 0 \end{array}$$

   iv. $$\begin{array}{rcrcr} 3x_1 & + & 5x_2 & \geq & 24 \\ x_1 & - & x_2 & \leq & 0 \\ 2x_1 & + & x_2 & \geq & 2 \end{array}$$

   The first is ok.

Consider the problem of finding x_1 and x_2 to solve the linear programming problem $\max x_0$ subject to $(x_1, x_2) \in S$, where x_0 is a linear function of x_1 and x_2 that is not constant (so $x_0 = Ax_1 + Bx_2$ and at least one of A and B is not zero). Call this problem **P**.

(b) For which of the following specifications of x_0 does the problem **P** have a **unique** solution?

i. $x_0 = x_1$

ii. $x_0 = x_1 - x_2$

iii. $x_0 = 5x_1 - 3x_2$

iv. $x_0 = -2x_1 - x_2$

The first and third are correct. The other have multiple solutions. Note that the last one is the same as minimizing $2x_1 + x_2$.

(c) Suppose that x_0 is a function with the property that $(3,3)$ solves **P**. For which of the following functions y_0 **must** it be the case that $(3,3)$ solves $\max y_0$ subject to $(x_1, x_2) \in S$?

i. $y_0 = x_0 + x_1$

ii. $y_0 = x_0 + x_2$

iii. $y_0 = 5x_0$

iv. $y_0 = -x_0$

The first and third are right. The third changes nothing. The first tilts the objective function to make $(3,3)$ even more attractive. The second could cause a shift to $(-2,6)$. The last changes the direction of increase and will move the solution (unless $x_0 = 0$).

(d) For which of the following pairs of points is it possible to find a non-constant x_0 such that both points solve **P**?

i. $(2,2)$ and $(\frac{2}{3}, \frac{2}{3})$

ii. $(4,4)$ and $(3,3)$

iii. $(-2,6)$ and $(2,2)$

iv. $(\frac{2}{3}, \frac{2}{3})$ and $(-2,6)$

The first and last are right. You want two points on the same edge. $(4,4)$ is not feasible and $(-2,6)$ and $(2,2)$ are not on the same edge.

4. (a) If a linear programming problem has more variables than constraints it is feasible.

   False. Not close.

   (b) If a linear programming problem is unbounded, then it will continue to be unbounded if the objective function changes.

   False. It is possible to change the objective function so that the problem is bounded (in fact, if the objective function is constant, then any feasible linear programming problem has a solution).

   The next six parts refer to the linear programming problem (P) written in the form:

$$\max c \cdot x \text{ subject to } Ax \leq b, x \geq 0$$

   and its dual (D):

$$\min b \cdot y \text{ subject to } yA \geq c, y \geq 0$$

   (c) If (P) has a unique solution, then its dual has a solution.

   True. This is a consequence of the duality theorem.

   (d) If x^* is a solution to (P), then x^* will be a solution to

$$\max rc \cdot x \text{ subject to } Ax \leq b, x \geq 0 \text{ for any } r.$$

   False. This would be true if $r \geq 0$, but when $r < 0$ the problem changes.

   (e) If (P) has a solution and $\bar{b} \geq b$, then

$$\max c \cdot x \text{ subject to } Ax \leq \bar{b}, x \geq 0$$

   has a solution.

   Yes. The second problem is feasible (since the solution to (P) satisfies the constraints). Moreover, its dual is feasible (because it has the same feasible set as (D) and (D) is feasible by the duality theorem). Hence the new problem must have a solution by the duality theorem.

   (f) If (D) has a solution, then the problem:

$$\max c \cdot x \text{ subject to } Ax \leq b/4, x \geq 0$$

   is feasible.

   Yes. Dividing the objective function of (D) by a positive constant won't change the solution of (D), so (P) will still have a solution (if x^* solves (P), then $x^*/4$ solves the new problem).

   (g) If (P) has a solution, x^*, then there exists a solution to (D), y^*, such that $b \cdot y^* = y^* A x^*$.

   True. This is a consequence of the duality theorem.

   (h) If (P) has a solution and $c' \geq c$, then

$$\max c' \cdot x \text{ subject to } Ax \leq b, x \geq 0$$

   has a solution. This is false. The new problem is still feasible, but it may be unbounded (if the change of leads to (D) being infeasible). One example is changing $\max -x$ subject to $-x \leq -1$, $x \geq 0$, to $\max x$ subject to $-x \leq -1$, $x \geq 0$.

5. A local restaurant makes three different kinds of burger. A classic cheeseburger uses one-quarter pound of ground beef and slice of cheese. A turkey burger uses one quarter pound of ground turkey. A double cheese burger uses one-half of a pound of ground beef and one slice of cheese. In addition, each type of burger requires a bun. The restaurant can sell a classic cheese burger for \$4.00, a turkey burger for \$5.50, and a double (cheese) burgers for \$8.00. Each day the restaurant has available 800 pounds of ground beef, 500 pounds of ground turkey, 2500 buns, and 2000 slices of cheese. The manager insists that the restaurant make at least 100 double cheese burgers every day. The restaurant wishes to know how many burgers of each type to produce in order to maximize profits subject to the constraints above. In order to formulate the problem, I defined the variables:

   x_1 = number of cheese burgers produced.

   x_2 = number of turkey burgers produced.

   x_3 = number of double burgers produced.

   The problem is then: find values for x_1, x_2, and x_3 to solve:

$$\begin{array}{lccccccc}
\max & 4x_1 & + & 5.5x_2 & + & 8x_3 & & \\
\text{subject to} & & & \frac{x_2}{4} & & & \leq & 500 \\
 & x_1 & & & + & x_3 & \leq & 2000 \\
 & \frac{x_1}{4} & & & + & \frac{x_3}{2} & \leq & 800 \\
 & x_1 & + & x_2 & + & x_3 & \leq & 2500 \\
 & & & & & x_3 & \geq & 100
\end{array}$$

$$x \geq 0.$$

   I solved this problem using Excel. The output follows this problem. Use the output to answer the questions on the next page. Answer the questions independently (so that a change described in one part applies only to that part). You must justify your answers by providing brief (but complete) descriptions of how you arrived at them.

   The two forms differed: on one the price of classic burgers was \$4, on the other \$7.

   (a) What is the restaurant's profit maximizing output? How much does it earn?

   0 cheese, 900 turkey, and 1600 double burgers, with profit 17750. Alternate form:

   800 cheese, 500 turkey, and 1200 double with profit 17950.

   (b) What is the most that the restaurant would be willing to pay for an additional pound of ground beef?

   \$5 (shadow price)

   Alternate form \$4

   (c) What is the most that the restaurant would pay for another slice of cheese?

   \$0 (it has left over cheese)

   Alternate: \$.5

   (d) Would the restaurant produce more double burgers if it could sell them for \$ 9 each?

   No. It is already producing as many as it can (the allowable increase on the coefficient is infinite so an increase in the coefficient does not change the solution).

   Alternative form: Yes. This is outside the allowable increase for the coefficient of double burgers.

(e) How much would the profits of the restaurant change if received 100 extra buns?

$550. (This is the shadow price of buns multiplied by 100.) Notice that 100 is in the allowable range. What happens is that the restaurant will sell 100 more turkey burgers. This is the answer for both forms.

(f) An employee suggests making Hawaiian burgers by replacing the cheese on a classic burger with a slice of pineapple. Pineapples cost 50 cents per slice. Would the restaurant want to put Hawaiian burgers on the menu if it could sell them for $7.50 each?

The cost of the ingredients of the Hawaiian burger are (from shadow prices) $5.50 for the bun and $2.50 for the meat. Since the pineapple costs 50 cents this means that it would need $8.00 to break even. So the answer is no.

On the other form, this is profitable at $7.50 because it saves $.50 per burger in cheese and the shadow price of beef is $4.00.

(g) The restaurant determines that it could sell a low carb cheeseburger that contains two slices of cheese, one-third of a pound of ground beef, but does not use a bun. What is the lowest price the restaurant could charge for this kind of burger (without lowering its profits)?

The restaurant thinks that cheese is free (it has an excess supply) so the only costly ingredient is beef, at $5 per pound. Hence the low carb burger is profitable at any price above $1.66 each.

On the alternative form the shadow price of beef is $4 and of cheese is $.5, so the store would need to get more than $2.33 to make a profit ($1 for two slices of cheese and $4/3 for the beef.