Econ 172A, W2002: Midterm Examination II, Possible Solutions

Comments Everyone received full credit for problems 3 a (vi) and b (vii) (I left out the * on the variables y_1^* and y^* in the statement of these problems).

1. (a) What is the solution to Professor Foster's problem and how much does he earn?

   He uses 56.25 acres for soybeans and raises 23.75 cows and earns $1000 * \$23.75 + 500 * \$56.25 = \$23,750 + \$28,125 = \$51,875$.

   (b) Professor Peters offers to buy 20 acres of Foster's land for $20 per acre. Should Professor Foster accept the offer? Why or why not?

   Foster has surplus land, so he will gladly accept the offer. Since the allowable decrease in land is greater than 20, he'll sell all twenty acres. He'll increase earnings by $400.

   (c) Professor Foster's son Andrew spends $10,000 of the investment funds on skateboard equipment (leaving only $30,000 to invest). How does this change the production plan and profits?

   There is surplus investment money of $11,500, so this change does not influence the solution or the value.

   (d) Andrew's skills as a skeleton racer permit him to earn $7 per hour during the winter as an instructor (raising the payment of surplus winter labor to $7 per hour). How does this change the production plan and profits?

   This change raises the objective function coefficient of unused winter labor from 5 to 7, which is more than the allowable increase of 1.25. Consequently the solution changes. Presumably Foster will rearrange production so that fewer labor hours are used in the winter, permitting Andrew to work as a skeleton instructor. Profits will go up, but the exact production plan requires solving another linear programming problem.

   (e) Suppose that it is possible to grow artichokes on Professor Foster's farm. For each acre of land devoted to growing artichokes, he must invest $100, use 30 person-hours of winter labor and 40 person-hours of summer labor. What is the minimum net annual cash income from artichokes needed to make it profitable for Foster to grow this crop?

   The resources for artichoke production are land, investment, 30 hours of winter labor, and 40 hours of summer labor. The first two things are in excess supply. The last two things are valued by their shadow prices of $6.25 and $7.50 per hour respectively. So the resources cost $187.50 + $300 = $487.50, so Foster must earn at least $487.50 per acre of artichokes in order to make production worthwhile.

   (f) Foster decides to quit his job teaching economics and devote more time to his farm. This increases the amount of winter labor available by 500 person hours (but does not change the amount of summer labor available). How does this change the production plan and profits?

   Winter labor increases by 500, this is in the allowable range. Production plan (soybeans and cows are the only things produced in positive quantities) remains the same, profits increase by 500 times the dual variable associated with winter labor, $6.25: $3,125.

   (g) How high would the price of corn need to be before it is profitable for Foster to grow corn?

   It must go up by $31.25 (to $781.25), since 31.25 is the allowable increase for the objective function coefficient of corn.

(h) Hogs require an investment of $1,000 each, require 10 acres of land, and require 100 person-hours of labor in both winter and summer. Would Foster raise hogs if he they produced an annual cash income of $1,200 each? Why or why not?

The value of the inputs is $1,375 (100 times the sum of the shadow prices of summer and winter labor), so Foster won't raise hogs if they only produce $1,200 per hog.

(i) Foster's cows gain so much weight that they start taking up more space in the barn. Suppose that the barn can only hold 20 cows. What can you say about Foster's production plan and profits?

Currently Foster is raising 23.75 cows. The change makes the barn-capacity constraint bind. It is not clear what Foster will do, but profits will go down. (They will not go down by more than 3.75*($1000 - $5*100 - $6*50) = $75; the first term is the price of cows, the second and third terms are lower bounds for what Foster would earn from the labor used in cow raising.

(j) How would the solution and value to Professor Foster's problem change if he were required to raise at least one hen on the farm?

This change adds a constraint to the problem. Since the constraint is not satisfied by the solution, profits will decrease. You can tell that profits will decrease by 1, because -1 is the reduced cost of hens. (Stated differently, the materials needed to raise one hen are worth one dollar more than what Foster can get selling the hens.)

2. Consider the following (imaginary) description of my bread-baking problem: I can make three different kinds of bread. A loaf of whole wheat bread uses one pound of whole wheat flour and an ounce of yeast. A loaf of oatmeal-rye bread uses three quarters of a pound of white flour, one quarter pound of rye flour, one quarter pound of oatmeal, and an ounce of yeast. A loaf of white bread uses three quarters of a pound of white flour and two ounces of yeast. I can sell a loaf of whole wheat bread for $2.00, a loaf of oatmeal-rye bread for $2.50, and a loaf of white bread for $1.50. Each day I have available 120 pounds of whole wheat flour, 100 pounds of white flour, 50 pounds of rye flour, 30 pounds of oatmeal, and 140 ounces of yeast. In addition, my ovens are able to bake at most 125 loaves each day. I want to know how many loaves of each type of bread to produce in order to maximize profits subject to the constraints above.

Define the following variables:

x_{wheat} = the number of loaves of whole wheat bread produced.

x_O = the number of loaves of oatmeal-rye bread produced.

x_{white} = the number of loaves of white bread produced.

y_{wheat} = the number of pounds of wheat flour used.

y_{white} = the number of pounds of white flour used.

y_O = the number of pounds of oatmeal used.

y_r = the number of pounds of rye flour used.

y_y = the number of ounces of yeast used.

(a) Which of the following constraints is appropriate for the problem? If a constraint is appropriate, explain how it relates to the problem description above. If a constraint is not appropriate, explain what the constraint says and why it does not apply to the problem.

i. $y_{white} = x_O + \frac{3}{4}x_{white}$
This constraint is appropriate. It states that the amount of white flour used can be computed by adding the white flour in the two kinds of bread that use white flour.

ii. $x_{white} = 2y_y + \frac{3}{4}y_{white}$
This constraint does not make sense. Knowing the amount of yeast and white flour used does not enable you to figure out how many loaves of white bread are produced.

iii. $y_y \leq 140$
This constraint is appropriate. It states that you have 140 ounces of yeast available.

(b) Using the variable definitions above, write an expression for the objective function of this problem.

$$\max 2x_{wheat} + 2.5x_O + 1.5x_{white}$$

(c) Using the variable definitions above, write a linear constraint that guarantees that at least 50% percent of the loaves produced are white bread.

$$x_{white} \geq .5x_{wheat} + .5x_O + .5x_{white}$$

(you can rearrange the terms if you wish). The left hand side is the number of loaves of white bread, the right hand side is the total number of loaves of bread (multiplied by .5).

(d) Using the variable definitions above, write a linear constraint that guarantees that at least as much white flour is used in the production of white bread as in the production of oat-rye bread.

$$.75x_{white} \geq x_O$$

3. Suppose that you solve a pair of linear programming problems, a primal, (P):

$$\max c \cdot x \text{ subject to } Ax \leq b, x \geq 0$$

and its dual, (D),

$$\min b \cdot y \text{ subject to } yA \geq c, y \geq 0.$$

Assume that both problems have at least two constraints (in addition to the non-negativity constraints), x^* is the solution to (P), and y^* is the solution to (D).

Decide whether each statement below is always true, sometimes true, or never true (under the stated conditions of the problem). That is, for each statement below, write "always," if the statement is true; "sometimes," if there exists problems (P) and (D) above, with solutions x^* and y^*, such that the statement is true **AND** (different) problems and solutions such that the statement is false; and write "never" if the statement is never true.

(a) Assume that each component of x^* is positive, $y_1^* > 0$, but all of the other components of $y^* = 0$.

i. The first constraint in the primal is binding.
This must be true by complementary slackness.

ii. The second constraint in the primal is binding.
This may or may not be true.

iii. The first constraint in the dual is binding.
This must be true by complementary slackness.

iv. The second constraint in the dual is not binding.
This cannot be true because it violates complementary slackness.

v. The value of the primal is $b_1 y_1^*$.
This is true. $b_1 y_1^*$ is the value of the dual and the value of the dual must equal the value of the primal by the duality theorem.

vi. $x_1^* y_1^* > 0$.
This is true (I assumed that both x_1 and y_1 are positive).

(b) Assume that the odd components of x^* (x_1, x_3, ...) are positive; the even components of x^* (x_2, x_4, ...) are zero; the first two constraints of (P) are binding, the other constraints of (P) are not binding.

i. The odd constraints of (P) are binding.
You do not have enough information to answer this question. It is sometimes true.

ii. At most two of the components of x^* are positive.
This is sometimes true. It depends on how many constraints there are in the problem.

iii. Exactly two of the components of x^* are positive.
This may be true. It depends on how many constraints there are in the problem.

iv. The first constraint of the dual is binding.
This must be true since $x_1^* > 0$.

v. The second constraint of the dual is binding.
This may be true (but it need not be true).

vi. $y_1^* > 0$.
This may be true (since the first constraint of the primal is zero), but it is not necessarily true.

vii. The odd components of y^* are zero.
This may be true, but it is not necessarily true.