Econ 172A, W2002: Final Examination, Solutions

Comments. Naturally, the answers to the first question were perfect. I was impressed. On the second question, people did well on the first part, but had trouble with the second part (you had to say something about how to solve the problem when there were more items than aisles). Most people did well on the third question (except for the final part). People had trouble with parts f and g of the fourth question. Only a very few got full credit on the fifth question. Most had trouble completely describing the constraints involving M and S.

1. Note: the other form of the question reversed the second and third rows and columns of the table. First, multiply payoffs by -1 to obtain a minimization problem:

	1	2	3	4
Soup	-10	-6	-12	-8
Bread	-15	-18	-5	-11
Beans	-17	-10	-13	-16
Candy	-14	-16	-6	-12

add a constant to each row to make the minimum in each row zero:

	1	2	3	4
Soup	2	6	0	4
Bread	3	0	13	7
Beans	0	7	4	1
Candy	2	0	10	4

subtract 1 from the fourth column to get a zero in that column:

	1	2	3	4
Soup	2	|6	|0	3
Bread	3	|0	|13	6
Beans	0	|7	|4	0
Candy	2	|0	|10	3

subtract 2 from uncrossed and add 2 to double crossed cells:

	1	2	3	4
Soup	0	6	0	1
Bread	1	0	13	4
Beans	0	9	6	0
Candy	0	0	10	1

and identify a solution: Soup - Aisle 3, Bread - Aisle 2, Beans - Aisle 4, Candy - Aisle 1. The profit is $12 + 18 + 16 + 14 = 60$. (Because of the different labeling, on the other form the best assignment is: Soup - Aisle 2, Bread - Aisle 4, Beans - Aisle 3, Candy - Aisle 1.

1

The second part of the table asks you to solve the assignment problem when you add a fifth column with all entries zero (that is, don't sell the item) and a fifth row in which all entries are 20:

	1	2	3	4	Don't Sell
Soup	10	6	12	8	0
Bread	15	18	5	11	0
Beans	17	10	13	16	0
Candy	14	16	6	12	0
Magazine	20	20	20	20	0

Changing to a minimization

	1	2	3	4	Don't Sell
Soup	2	6	0	4	0
Bread	3	0	13	7	6
Beans	0	7	4	1	5
Candy	2	0	10	4	4
Magazine	0	0	0	0	8

This is an assignment problem, which you can solve using the Hungarian method.

2. More than two hundred years ago the French mathematician de Montmort proposed the following gift to his son. "I shall put a gold coin into either my right hand or my left hand. You will name a hand. If your guess is correct and the coin is in my right hand, you will get the gold coin (on the other form, three gold coins). If your guess is correct and the coin is in my left hand, you shall receive two gold coins. Otherwise, you will get nothing." Assume that de Montmort and his son play this game, that de Montmort wants to minimize the amount of gold that he expects to pay to his son and his son wants to maximize the amount of gold he wins.

   (a) Write the payoff matrix for this game. Clearly label the strategies and explain how you computed the payoffs.

	Left	Right
Left	2	0
Right	0	1

These are the payoffs to the son assuming that he chooses row and the father chooses columns. The father loses what the son wins. So, for example, if the father plays left and the son plays left, the son wins two gold coins. (On the other form, the 1 is changed to three.)

(b) Find the pure-strategy security levels for both players. Would a rational player use a pure strategy in this game? Explain.

The security level for the son is zero. The security level for the father is -1. That is, the son can guarantee a payoff of zero (no matter what he does), while the father can hold the son to a payoff of 1 (by playing right). There is a gap between these numbers, so at least one player can do better by playing a mixed strategy. (On the other form, the father's security level is -2.)

(c) Are there any dominated strategies in the game? Identify them and explain why they are dominated.

There are no dominated strategies.

(d) Find the mixed strategy security level of the game.

The mixed strategy security level for row is $\frac{2}{3}$, which he obtains by playing left with probability one third (and right with probability two thirds). Similarly, the mixed strategy security level for column is attained by playing left with probability one third. (On the other form, the son plays left with probability .6, right with probability .4; the father does the same; the value is 1.2.)

(e) What is the value of the game and what are the equilibrium strategies?

The strategies in the previous question are the equilibrium strategies. The value of the game (to the son) is his security level of $\frac{2}{3}$.

(f) Suppose now that the son is able to see which hand de Montmort hides the coin before the son decides which hand to guess. (De Montmort knows that his son knows where the coin will be hidden.) Write the payoff matrix for this version of the game. Answer questions (c) and (e) for this version of the game.

	Left	Right
Always Left	2	0
Always Right	0	1
Match	2	1
Mismatch	0	0

If the son can peek, then he has four strategies. If he matches, then he always wins. The amount that he wins depends on what his father

does. If he mismatches, then he never wins. Matching is a dominant strategy. The security level of the son is one, which is also the most that he can get if the father plays cautiously. So the value of the game is 1; the son always matches; the father plays right. (On the other form, the 1's in the right column become 3's; the son matches, the father plays left, and the value is 2.)

3. A company can produce two products. The table below summarizes the production technology. Each week, up to 400 units of raw material can be purchased at a cost of $1.50 per unit. The company employs four workers, who work 40 hours per week. The base salaries of the workers are considered a fixed cost and do not enter the computation. Workers are paid $6 per hour to work overtime. Each week, 320 hours of machine time are available.

	Product 1	Product 2
Selling Price	$15	$8
Labor Required	.75 hour	.5 hour
Machine Time Required	1.5 hour	.8 hour
Raw Material Required	2 units	1 unit

If the firm does not advertise, 50 units of product 1 and 60 units of product 2 will be demanded each week. Advertising can be used to stimulate demand. Each dollar spent on advertising Product 1 increases the demand for Product 1 by 10 units. Each dollar spent on advertising Product 2 increases the demand for Product 2 by 15 units. At most $100 can be spent on advertising.

To formulate the problem, I defined the following variables:

P_1 = the number of units of Product 1 produced each week.

P_2 = the number of units of Product 2 produced each week.

OT = the number of hours of overtime labor used each week.

RM = the number of units of raw material purchased each week.

A_1 = the amount (in dollars) spent each week advertising Product 1.

A_2 = the amount (in dollars) spent each week advertising Product 2.

The firm's optimization problem is then:

$$
\begin{array}{lrcrcrcrcrcrcrl}
\max & 15P_1 & + & 8P_2 & - & 6OT & - & 1.5RM & - & A_1 & - & A_2 & & & \\
\text{subject to} & P_1 & & & & & & & - & 10A_1 & & & \leq & 50 \\
& & & P_2 & & & & & & & - & 15A_2 & \leq & 60 \\
& .75P_1 & + & .5P_2 & - & OT & & & & & & & \leq & 160 \\
& 2P_1 & + & P_2 & & & - & RM & & & & & \leq & 0 \\
& & & & & & & RM & & & & & \leq & 400 \\
& & & & & & & & & A_1 & + & A_2 & \leq & 100 \\
& 1.5P_1 & + & .8P_2 & & & & & & & & & \leq & 320
\end{array}
$$

$$P_1, P_2, OT, RM, A_1, A_2 \geq 0.$$

I solved this problem using Excel. The output follows this problem. Use the output to answer the questions below. Answer the questions independently (so that a change described in one part applies only to that part).

Answer as many questions below as you can using the available information. If there is some question that you cannot answer using the output that I have provided, explain why not and say as much as you can using the available information.

(a) If overtime costs $4 per hour, would the company use it?

No. The company doesn't use it now and lowering the cost by $2 stays within the allowable range.

(b) If each unit of Product 1 sold for $15.50 would the current basis remain optimal? What would be the new solution?

Yes, because an increase of $.50 remains in the allowable range. The production plan doesn't change. Profit increases by $.5 \times 160 = $80.

(c) What is the most that the company would be willing to pay for another unit of raw material?

A total of $6 ($4.50 more than it normally does).

(d) How much would the company be willing to pay for another hour of machine time?

Nothing. There is slack in the constraint (and dual variable is zero).

(e) If each worker were required (as part of the regular work week) to work 45 hours per week, what would the company's profit be? (That is, assume that the company gets 45 hours per week from a worker without paying overtime.)

This increases labor supply to 180 hours (from 160). This is within the allowable range. Profit goes up by 20 times the shadow price of the labor constraint. So profit goes up by 77.333.

(f) How would the solution of the problem change if you could not use more than 20 hours of machine time per week in the production of the first product?

This imposes a new constraint on the problem: $1.5P_1 \leq 20$. This constraint is not satisfied by the original solution. You can't be sure what happens without solving the modified problem, but profits will go down. Profits won't go down by more than what would happen if the production of product one dropped from 160 to 13.33 (the most P_1 you could make with available machine time). You can use shadow prices to get an approximation of how much this would lower profits, but since the reduction will be outside of the allowable range for many of the constraints, the approximation won't be too accurate.

(g) How would the firm's profits change if advertising for Product 1 was free?

Increasing the coefficient of advertising in the objective function by one (in order to make advertising free) is within the allowable range. Hence the basis does not change. Profits go up by one dollar times the number of units of A_1 purchased. That is, profits go up by $11.

(h) Suppose that the technology was improved so that only .5 hours of labor were needed to produce one unit of Product 1. How would the answer change? I do not know how to answer this without solving

the problem again, but you can see that the company is using all of its available labor and that the shadow price on labor is $3.867, with an allowable increase of 27.5. The change frees up 40 hours of labor (.25 hours saved for each of 160 units). Hence the company gains at least $27.5 * \$3.867 = \$106.33$.

4. The California Cheese Company produces two cheese spreads by blending mild cheddar cheese with sharp cheddar cheese. The cheese spreads are sold in one pound containers. There are two blends. Regular blend contains 80% mild cheddar and 20% sharp cheddar. The tangy blend contains 60% mild cheddar and 40% sharp cheddar. The company can buy as much as 8100 pounds of mild cheddar cheese for $1.20 per pound and up to 3000 pounds of sharp cheddar cheese for $1.40 per pound. The cost to blend and package the cheese spreads is $.20 per one-pound container. (This cost does not include the cost of the cheese.) Each container of the Regular blend sells for $2.60 and each container of Tangy sells for $2.20. The California Cheese Company wants to determine how many containers of each blend to produce. Its goal is to maximize profits.

In order to formulate its problem, the company defines the following variables:

$R =$ number of containers of regular blend produced.

$T =$ number of containers of tangy blend produced.

$M =$ number of pounds of mild cheddar used.

$S =$ number of pounds of sharp cheddar used.

Use these variables and the information about the problem given above to provide the information requested. (On the other form, the only change was that tangy was 70% mild and 30% sharp.)

(a) Write an expression for the total cost of the company's production plan.

$$1.20M + 1.40S + .2(R + T)$$

The first and second terms are the costs of the input cheeses, while the third term is the blending and packaging cost.

(b) Write an expression for the total revenue of the company's production plan.

$$2.60R + 2.20T$$

(c) Write expressions that correctly describe the availability of mild and sharp cheddar cheeses.

$$M \leq 8100, S \leq 3000$$

(d) Write down any other constraints needed to completely describe the optimization problem of the cheese company.

$$M = .8R + .6T, S = .2R + .4T$$

and
$$S, T, M, R \geq 0$$

(on the other form, the coefficient of T in the first constraint is .7 and in the second constraint .3).

(e) Write the linear programming problem that describes the company's optimization problem. Use only the expressions you wrote as answers to the previous parts of the question.

$$
\begin{array}{lrcrcrcrcr}
\text{max} & 1.20M & + & 1.40S & - & 2.4R & - & 2T & & \\
\text{subject to} & M & & & - & .8R & - & .6T & = & 0 \\
& & & S & - & .2R & - & .4T & = & 0 \\
& M & & & & & & & \leq & 8100 \\
& & & S & & & & & \leq & 3000 \\
& M & & S & & R & & T & \geq & 0
\end{array}
$$

5. The responses are i, ii, iii, iv on this form correspond to iv, i, ii, iii on the other form.

(a) The inequalities below describe two-dimensional sets. Which of these sets can be described by linear inequalities?

    i. $x + y^2 \leq 4$. No (it looks like a parabola).

    ii. $x + y \leq 4$. Yes.

    iii. $|x + y| \leq 4$. Yes: $x + y \leq 4, x + y \geq -4$

    iv. $\frac{x}{x+y+2} \leq 4, x \geq 0, y \geq 0$. Yes: $x \leq 4x + 4y + 8$ (or $-3x - 4y \leq 8$).

(b) Which of the following must be true in any valid simplex algorithm array with at least one constraint?

    i. All entries in the value column (excluding the value of x_0) are non-negative. Yes. This is the feasibility condition.

    ii. There is at least one non-negative number in each column. No.

    iii. There is at least one zero in each row. Yes. This is the basis condition.

    iv. There is at least one negative number in row 0. No (not when the problem is solved).

(c) Which of the statements below are true statements about following linear programming problem (P) or its dual (D):

$$
\begin{array}{rrcrcrcrcr}
\max & 15x_1 & + & 6x_2 & + & 9x_3 & + & 2x_4 & & \\
\text{subject to} & 2x_1 & + & x_2 & + & 5x_3 & + & 6x_4 & \leq & 20 \\
& 3x_1 & + & x_2 & + & 3x_3 & + & 3x_4 & \leq & 24 \\
& 7x_1 & & & & & + & x_4 & \leq & 70 \\
& & & & & & & x & \geq & 0
\end{array}
$$

    i. $x^* = (4, 12, 0, 0)$ is a solution to (P). Yes. You can verify this by complementary slackness.

    ii. $x^* = (10, 6, -4, 0)$ is a solution to (P). No. It violated the non-negativity constraint.

    iii. $y^* = (3, 3, 0)$ is a solution to (D). Yes. Complementary Slackness.

    iv. If the right-hand side of the third constraint increased from 70 to 85, the solution to (P) would not change. Yes. This constraint is not binding in the solution.

(d) Consider the linear programming problem:

$$\max c \cdot x \text{ subject to } Ax \leq b, x \geq 0.$$

Assume that all entries in A, b, and c are whole numbers and that the problem has a solution x^*. By a whole number solution, I mean a solution x^{**} with the property that every component of x^{**} is a whole number.

    i. The problem must have a whole number solution. No.

ii. The problem must have a whole number solution provided that all of the entries in A, b, and c are equal to -1, 0, or 1. No.

iii. The problem must have a whole number solution provided that the dual of the problem has a whole number solution. No.

iv. If the problem has a whole number solution, then the value of the problem is a whole number. Yes (the value of the problem is obtained by multiplying whole numbers together and adding them).

The transportation and assignment problems do have the property that whole number costs imply whole number solutions.