

Econ 172A, F2003: Midterm Examination II, Possible Solutions

**Comments:** Mean and median both 69

1. There were two mistakes that showed a lack of understanding of complementary slackness. One mistake was large, the other small. The large mistake: CS permits you to infer that if  $x_j > 0$ , then the  $j^{th}$  dual constraint must bind. Several papers used some other method to determine which constraints in the dual had to bind. The small mistake: If a constraint binds, then CS does not permit you to infer that the corresponding variable in the dual is positive (it may be zero). A few people found that the  $x$  in part (i) solved the problem and then stopped. This is not a sufficient answer. You need to supply some reason why the other choices for  $x$  are not also solutions (it suffices to check that they give different values). Finally, if you correctly follow CS rules, then you are guaranteed to come up with  $y$  that yields the same value in (D) as  $x$  does in (P) (provided that it is possible to solve for  $y$ ). Checking that values are equal confirm that you did your computation correctly, but does not prove that  $x$  is a solution, for this you must check that  $y$  satisfies **all** dual constraints. On this problem you received 3 points for part (a); 15 points for b(i) and c; 5 points for b(ii); and 7 points for b(iii). After some mistakes (using incorrect inferences that b(i) was not a solution), we used a more generous scale to reward knowledge of some aspects of CS.
2. (a) (10 points) It was important to say more that “ $T = \text{toys.}$ ” Variables count things. Units were far more important in the last part of this question, but you lost two points for failing to make that clear here.  
(b) (4 points) There were different possible ways to answer (b), (d), and (e), but some explanations were incorrect.  
(c) (3 points)  
(d) (4 points)  
(e) (4 points)  
(f) (10 points) Several papers had bits and pieces of the buy out story, but missed enough so that the entire answer made little sense. It is essential to recognize that the dual variables represent prices of the inputs (so that they are denominated in “dollars per unit”) and not quantities. The dual tells you that the value of ingredients – when ingredients are “correctly” priced – is equal to the value of outputs.
3. Grading this part was straightforward: you received .5 point for each of the blanks (total score rounded up); for the rest: 1, 1, 2, 3, 3, 3, 3 (respectively). It was difficult to answer some of the last parts if you incorrectly completed the table. We tried to take that into account. Many answers showed an incomplete understanding of sensitivity information and, in particular, the meaning and uses of dual variables. Please read the answer carefully.

Note: In form (b),  $x_1$  and  $x_3$  are reversed in the first question and a few of the numbers (but not the story) are changed in the second question.

1. Consider the linear programming problem:  
Find  $x_1$ ,  $x_2$  and  $x_3$  to solve:

$$\begin{array}{rcll}
\max & x_1 & + & 3x_2 & + & 4x_3 & & \\
\text{subject to} & x_1 & + & x_2 & + & x_3 & \leq & 5 \\
& 2x_1 & - & x_2 & - & 4x_3 & \leq & 4 \\
& & & x_2 & + & 2x_3 & \leq & 3 \\
& & & & & x & \geq & 0
\end{array}$$

(a) Write the dual of the problem.

$$\begin{array}{rcll}
\min & 5y_1 & + & 4y_2 & + & 3y_3 & & \\
\text{subject to} & y_1 & + & 2y_2 & + & & \geq & 1 \\
& y_1 & - & y_2 & + & y_3 & \geq & 3 \\
& y_1 & - & 4y_2 & + & 2y_3 & \geq & 4 \\
& & & & & y & \geq & 0
\end{array}$$

(b) Use Complementary Slackness to determine which of the vectors  $(x_1, x_2, x_3)$  below is a solution to the problem.

i.  $(x_1, x_2, x_3) = (2, 3, 0)$

$(2, 3, 0)$  is feasible for the primal; the first and last constraints bind; the second constraint does not. Hence, if it solves the primal, the solution to the dual must satisfy the first two dual constraints as equations and  $y_2 = 0$ . To find  $y_1$  and  $y_3$ , solve the first two dual constraints as equations (using  $y_2 = 0$ ):  $y_1 = 1$  and  $y_1 + y_3 = 3$ . Hence  $y_3 = 2$ . Hence if  $(2, 3, 0)$  solves the Primal, then  $(1, 0, 2)$  solves the Dual. To complete the answer just check that  $(1, 0, 2)$  is feasible for the Dual. It obviously satisfies non-negativity and the first two constraints. You can check that the third constraint also holds. Hence  $(2, 3, 0)$  does solve the Primal and  $(1, 0, 2)$  does solve the dual.

ii.  $(x_1, x_2, x_3) = (4, 0, 1)$

$(4, 0, 1)$  is not feasible for the primal (it does not satisfy the second constraint). Hence it cannot solve the primal.

iii.  $(x_1, x_2, x_3) = (3, 1, 1)$

$(3, 1, 1)$  is feasible for the primal; the first and third constraints bind; the second constraint does not. Hence, if it solves the primal, the solution to the dual must satisfy the all constraints as equations and  $y_2 = 0$ . To find  $y_1$  and  $y_3$ , solve the dual constraints as equations (using  $y_2 = 0$ ):  $y_1 = 1$ ,  $y_1 - y_2 + y_3 = 3$ , and  $y_1 + 2y_3 = 4$ . You can see that this is impossible, so  $(3, 1, 1)$  can't solve the primal.

(c) Use your answer to the previous question to find a solution to the dual.

I obtained the solution to the dual in the answer to question 2a. (Notice that once you discover that  $(2, 3, 0)$  solves the primal, you can rule out the other two possible answers because they give different values for the primal objective function.)

2. BENCO makes three types of toys: trains, trucks, and cars. Trains earn a profit of \$3 each; trucks earn \$2 each; and cars earn \$5 each. The toys are made from wood, paint, and labor. To make a train you need one minute of labor; three ounces of paint; and 1 square foot of wood. To make a truck you need two minutes of labor; one ounce of paint; and four square feet of wood. To make a car you need one minute of labor; two ounces of paint; and one square foot of wood. BENCO has available 430 minutes of labor; 460 ounces of paint; and 420 square feet of wood. BENCO wishes to determine what to produce in order to maximize profits.

(a) Formulate a linear programming problem that will determine BENCO's profit maximizing production plan. Your formulation should include: a definition of variables (including units); an algebraic expression for the objective function; and an algebraic expression for the constraints.

Define the variables:  $R$ ,  $T$  and  $C$  to be the number of trains, trucks, and cars produced respectively (sorry, train and truck both start with "t").

Find the objective function:

$$\max 3R + 2T + 5C.$$

Write down the constraints:

Labor:

$$R + 2T + C \leq 430$$

Paint:

$$3R + T + 2C \leq 460$$

Wood:

$$R + 4T + C \leq 420$$

Summarizing (and adding the obvious non-negativity constraint) the problem is to find  $R, T, C$  to solve:

$$\begin{array}{rllll} \max & 3R & + & 2T & + & 5C \\ \text{subject to} & R & + & 2T & + & C & \leq & 430 \\ & 3R & + & T & + & 2C & \leq & 460 \\ & R & + & 4T & + & C & \leq & 420 \\ & R & & T & & C & \geq & 0 \end{array}$$

(b) Is the problem you wrote in part (a) feasible? If it is, explain why it is. If it is not, explain why it is not. If you cannot tell, explain what additional information you would need to know in order to determine whether it is feasible.

The problem is feasible. For example, if you set  $T = R = C = 0$ , then you satisfy all of the constraints.

(c) Write the dual of the problem you found in part (a).

$$\begin{array}{rllll}
\min & 430L & + & 460P & + & 420W \\
\text{subject to} & L & + & 3P & + & W & \geq & 3 \\
& 2L & + & P & + & 4W & \geq & 2 \\
& L & + & 2P & + & W & \geq & 5 \\
& L & & P & & W & \geq & 0
\end{array}$$

(I picked  $L, P, W$  as variable names in anticipation of the interpretation question.)

- (d) Is the problem you wrote in part (c) feasible? If it is, explain why it is. If it is not, explain why it is not. If you cannot tell, explain what additional information you would need to know in order to determine whether it is feasible.

Yes. For choosing the variables to be large enough you can satisfy all of the constraints. Certainly  $L$  bigger than 5 (other variables zero) satisfies the constraints.

- (e) Does the problem in part (a) have a solution? If it does, explain why. (It is not necessary to solve the problem.) If it does not, explain why not. If you cannot tell, explain what additional information you would need to know whether they problem had a solution.

Yes, by the duality theorem because both primal and dual are feasible.

- (f) Interpret the dual that you found in part (c). Your interpretation should define the dual variables in words (and provide units for the dual variables) and explain what the dual objective function and constraints mean in your problem.

Interpret the dual variables are prices:  $L$  is the price of one minute of labor;  $P$  is the price of one ounce of paint;  $W$  is the price of one square foot of wood. These prices should be thought of as the value to the owner of BENCO of these resources.

The dual is a “buy out” problem. Someone tries to buy the resources of BENCO. In order to do so, she sets prices for all of the inputs (labor, paint, and wood) so that the toy maker prefers to sell the inputs rather than turn them into toys. The constraints all have the interpretation that the profit available from the toy is no greater than the value of its inputs. Specifically, the first constraint states that the profit from a train (\$3) is less than or equal to  $L + 3P + W$ , which is the value of the materials used to make the train. The other two constraints are similar. The non-negativity constraint says that these prices must be non-negative. The objective is to minimize what it costs to buy all of the ingredients.

3. I solved a linear programming problem written in the form:

$$\max c \cdot x \text{ subject to } Ax \leq b, x \geq 0.$$

Attached find the Excel Answer and Sensitivity report. (I deleted some irrelevant information.) In these reports, I replaced several values with letters ((a) through (ii)). Using the information in the table, replace as many question marks as possible with the correct information. You need not justify these answers (simply write the answers in the appropriate spaces, next to the question mark). If you do not have enough information to figure out one or more of the values, write "NOT ENOUGH INFORMATION" next to the question marks.

In addition to completing the tables, please answer the following questions. For these questions, short justifications are required.

- (a) How many variables are in the original problem?  
Four.
- (b) How many variables are in the dual?  
Three (since there are three primal constraints).
- (c) What is the objective function of the original problem?  
 $x_1 + 2x_2 + 3x_3 + 4x_4$ .
- (d) Would the solution to the problem change if the coefficient of  $x_2$  in the objective function were decreased by 3 (and the rest of the problem remained unchanged)?  
No change to the solution, since the allowable decrease in  $3.275 > 3$ .
- (e) What would the value of the problem be if the right hand side constant on the first constraint were 224 (and the rest of the problem remained unchanged)?  
The right-hand side increases by 100, this is within the allowable range. The value goes up by 100 times the dual variable. The dual variable is equal to .225433526. Hence the value becomes 95.5433526.
- (f) What would the value of the problem be if the right hand side constant on the second constraint were 50 (and the rest of the problem remained unchanged)?  
The right-hand side increases by 7, this is within the allowable range. The value goes up by 7 times the dual variable. The dual variable is equal to .757225434. Hence the value becomes 78.300578038.
- (g) What would the solution to the problem change if the coefficient of  $x_1$  in the objective function was 5 (and the rest of the problem remained unchanged)?  
The solution would change because this increases the coefficient of  $x_1$  by 4, which is greater than the allowable increase of 3.156.