Econ 202a Spring 2000

Suggested Final Exam SOLUTIONS

1 Rational and Near Rational Wage Setting Firms

Each firm faces a commodity demand of $q^d = \frac{1}{n}\frac{M}{\overline{p}}\left(\frac{p}{\overline{p}}\right)^{-\beta}$, where $\beta > 1$. Labor productivity is given by $T(w) = -A + B\left(\frac{w}{w^R}\right)^{\alpha} + Cu$. So, the production function is $q^s = T(w)L$ and the cost function must take the form $C(q^s) = \frac{w}{T(w)}q^s$. Hence, the profit function can be written as

$$\Pi_i = p_i q_i^s - C(q_i^s) = \left(p_i - \frac{w_i}{T(w_i)}\right) q_i^d \qquad (1)$$

for a monopolist. There are two types of firms, rational and near-rational ones. Label them $i = r, nr$, respectively.

1.1 Price choice of firm i

Firm i chooses it price by maximizing (1) with respect to p_i. The first order condition for this problem is

$$q_i^d - \beta\left(p_i - \frac{w_i}{T(w_i)}\right)\frac{q_i^d}{p_i} = 0,$$

which is a sufficient condition since revenues are concave in p_i. Rearranging yields

$$p_i = \frac{\beta}{\beta - 1}\frac{w_i}{T(w_i)}. \qquad (2)$$

1.2 Wage choice of firm i

Similarly, maximizing (1) with respect to w_i yields the first order condition

$$-\left(\frac{T(w_i) - T'(w_i)w_i}{T(w_i)^2}\right) = 0$$

so that, in optimum, the *Solow condition*

$$w_i = \frac{T(w_i)}{T'(w_i)} \tag{3}$$

must be satisfied. By the definition of $T(\cdot)$, this implies that

$$w = \frac{-A + B\left(\frac{w_i}{w_i^R}\right)^{\alpha} + Cu}{\alpha B\left(\frac{w_i}{w_i^R}\right)^{\alpha}\frac{1}{w}}.$$

Equivalently,

$$w_i = \left(\frac{A - Cu}{(1-\alpha)B}\right)^{\frac{1}{\alpha}} w_i^R. \tag{4}$$

1.3 Relative profits

Given the facts that $w_r^R = w^R = \overline{w}_{-1}(1+\pi^e)$ for the rational firms and $w_{nr}^R = \overline{w}_{-1}$ for the near rational firms, the two wage choices are equal for $\pi = \pi^e = 0$:

$$w_r^R = w_{nr}^R = \left(\frac{A - Cu}{(1-\alpha)B}\right)^{\frac{1}{\alpha}} w^R \quad \text{if } \pi = \pi^e = 0.$$

Hence, the productivity levels must be equal, too: $T(w_r) = T(w_{nr})$ for $\pi = \pi^e = 0$. Then, however, price choices must be equal: $p_r = p_{nr}$. Therefore, we can conclude that profits Π_r and Π_{nr} must be identical if $\pi = \pi^e = 0$ and relative profits (the ratio of profits) must be equal to one: $\Pi_r/\Pi_{nr} = 1$.

Suppose these relative profits could be expressed as a function of inflation π. Then, by the arguments above, at zero inflation $\pi = \pi^e = 0$ the optimal choice of p_i and w_i are the same for both firms. Therefore, the change in profits is the same by the envelope theorem so that the ratio of profits remains at one after the change. This means, however, that the derivative of the ratio Π_r/Π_{nr} with respect to π must be zero.[1] This argument suffices to answer the question.

[1] In general,

$$\frac{\partial\left(\frac{\Pi_r}{\Pi_{nr}}\right)}{\partial \pi} = \frac{\frac{\partial \Pi_r}{\partial \pi}\Pi_{nr} - \frac{\partial \Pi_{nr}}{\partial \pi}\Pi_r}{(\Pi_{nr})^2} = 0$$

for $\Pi_r = \Pi_{nr}$ and $\partial \Pi_r/\partial \pi = \partial \Pi_{nr}/\partial \pi$.

The same results could, of course, also be derived explicitly. This was not required. For completeness, relative profits could be derived by noting that

$$\begin{aligned}\Pi_i &= \left(p_i - \frac{w_i}{T(w_i)}\right) q_i^d \\ &= \left[\frac{\beta}{\beta-1}\frac{w_i}{T(w_i)} - \frac{w_i}{T(w_i)}\right]\frac{1}{n}\frac{M}{\overline{p}}\left(\frac{p_i}{\overline{p}}\right)^{-\beta} \\ &= \frac{1}{\beta-1}\left(\frac{\beta-1}{\beta}\right)^{\beta}\frac{M}{n}\left(\frac{1}{\overline{p}}\right)^{1-\beta}\left(\frac{w_i}{T(w_i)}\right)^{1-\beta},\end{aligned}$$

and hence

$$\frac{\Pi_r}{\Pi_{nr}} = \left[\frac{\frac{w_r}{T(w_r)}}{\frac{w_{nr}}{T(w_{nr})}}\right]^{1-\beta} = \left[\frac{\frac{w_r^R}{w_{nr}^R}}{\frac{-A+B(w_r/w^R)^{\alpha}+Cu}{-A+B(w_{nr}/w^R)^{\alpha}+Cu}}\right]^{1-\beta}$$

by (4) and the fact that true productivity is given by $T(w_i) = -A + B\left(w_i/w^R\right)^{\alpha} + Cu$ for both firms (where $w^R = \overline{w}_{-1}(1+\pi^e)$). Setting $\pi = \pi^e = 0$ immediately yields $\Pi_r/\Pi_{nr} = 1$. Taking the derivative with respect to π and evaluating the resulting term at $\pi = \pi^e = 0$ yields zero.

1.4 Short-run *wage* Phillips curve

If we suppose that Φ denotes the share of near-rational firms, we can obtain the wage level at period 0 as

$$\begin{aligned}\overline{w}_0 &= \Phi w_{nr} + (1-\Phi)w_r \\ &= \left(\frac{A-Cu}{(1-\alpha)B}\right)^{\frac{1}{\alpha}}\left[\Phi\overline{w}_{-1} + (1-\Phi)\overline{w}_{-1}(1+\pi^e)\right]. \end{aligned} \tag{5}$$

Subtracting $\overline{w}_{-1}/\overline{w}_{-1}$ from both sides of (5) yields

$$\frac{\overline{w}_0 - \overline{w}_{-1}}{\overline{w}_{-1}} = \left(\frac{A-Cu}{(1-\alpha)B}\right)^{\frac{1}{\alpha}}\left[1 + (1-\Phi)\pi^e\right] - 1. \tag{6}$$

1.5 Long-run Phillips curve

Since $\pi = \pi^e = 0$ and $\pi = (\overline{w}_0 - \overline{w}_{-1})/\overline{w}_{-1}$ in the long-run, (6) implies

$$1 + \pi_{LR} = \overline{a}\left[1 + (1-\Phi)\pi_{LR}\right] \tag{7}$$

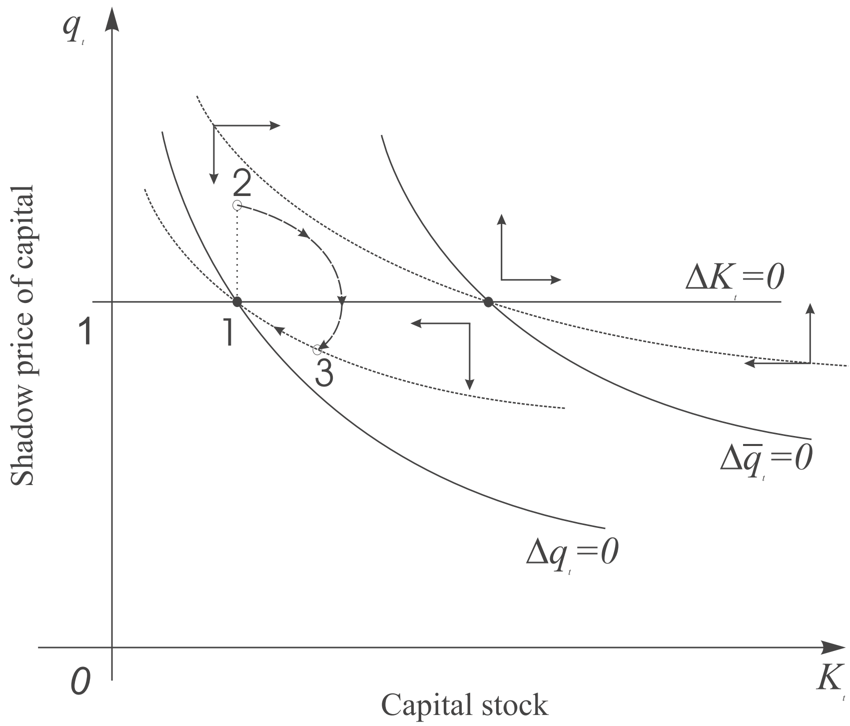

Figure 1: **Phase Diagram for Tobin's *q***

for $\overline{a} \equiv [(A - Cu)/(1-\alpha)B]^{\frac{1}{\alpha}}$. Thus,

$$\pi_{LR} = -\frac{1-\overline{a}}{1-\overline{a}(1-\Phi)}. \tag{8}$$

2 Temporary Technological Progress and Tobin's *q*

A temporary rise in the technology parameter A shifts the $\Delta q_t = 0$-schedule temporarily to the East. Thus, between time t_0, when productivity unexpectedly rises, and time $T > t_0$, when it expectedly falls back to its initial level, the new dynamic system with $\Delta\overline{q}_t = 0$ and $\Delta K_t = 0$ temporarily governs the dynamics of the capital stock and Tobin's q.

The capital stock K_{t_0} cannot adjust immediately and thus remains at its initial steady-state level for one period. However, the shadow price of capital, Tobin's q, has to jump to a new level which is such that the subsequent levels of both q_s and K_s $(s \geq t_0)$ will be back at a point on the initial saddle path. So, at t_0 Tobin's q jumps discretely to point 2 in figure 1 and it will reach point 3 exactly at date T. Between time t_0 and T, both variables q_s and K_s must follow the dynamics of the shifted system which take them first to the south east and then (once the $\Delta Kt = 0$-schedule is surpassed) to the south west of the system. Note that the shadow price must not change discontinuously (jump) at any time after t_0, otherwise the capital stock cannot have been chosen optimally. Between time t_0 and T, the capital stock first increases and then falls while Tobin's q keeps falling until T. Once time T is reached, the system is again governed by the 'old' $\Delta q_t = 0$ and $\Delta K_t = 0$-schedules and the capital stock continues to fall smoothly while the shadow price of capital q_s now continuously rises.

3 Central Bank with Superior Knowledge

3.1 Linear optimal monetary rule

Given aggregate supply

$$y_t = (p_t - \mathbb{E}_{t-1}[p_t]) + u_t, \tag{9}$$

and aggregate demand

$$m_t^d = p_t + \tilde{v}_t, \tag{10}$$

output can be written as

$$\begin{aligned} y_t &= m_t^s - \tilde{v}_t - \mathbb{E}_{t-1}[m_t^s - \tilde{v}_t] + u_t \\ &= m_t^s - \tilde{v}_t - \mathbb{E}_{t-1}[m_t^s] + \alpha u_{t-1} + \tilde{\epsilon}_t. \end{aligned} \tag{11}$$

The second step follows from the assumptions on the stochastic processes. Since the conditional expectations over the linear monetary rule $m_t^s = a + bu_{t-1} + c\tilde{\epsilon}_t + dv_{t-1}$ are $\mathbb{E}_{t-1}[m_t^s] = a + bu_{t-1} + d\,v_{t-1}$, it follows that

$$m_t^s - \mathbb{E}_{t-1}[m_t^s] = c\tilde{\epsilon}_t.$$

Hence, output can be written as

$$y_t = (c+1)\tilde{\epsilon}_t - \tilde{v}_t + \alpha u_{t-1}. \tag{12}$$

For $c = -1$, the disturbance of aggregate supply is completely offset by the monetary rule so that the variance of output will be minimized. The parameters a, b, d are irrelevant since their impact is entirely anticipated by the private sector. This arguments can be made formal by minimizing $\mathbb{V}(y_t) = \mathbb{E}[(c\tilde{\epsilon}_t - \tilde{v}_t + \alpha u_{t-1} + \tilde{\epsilon}_t)^2]$ with respect to a, b, c, d (a, b, d don't enter the minimization problem).

3.2 Non-linear optimal monetary rule

Allowing for higher order terms such as in a rule like $m_t^s = a + bu_{t-1} + c\tilde{\epsilon}_t + dv_{t-1} + b'(u_{t-1})^2 + c'(\tilde{\epsilon}_t)^2 + d'(v_{t-1})^2$ will result in minimizing $\mathbb{V}(y_t) = \mathbb{E}[(c\tilde{\epsilon}_t + c'(\tilde{\epsilon}_t)^2 - \tilde{v}_t + \alpha u_{t-1} + \tilde{\epsilon}_t)^2]$. Now neither a, b, d nor a', b', d' enter, and setting $c = -1$ is still optimal. Since the noise $\tilde{\epsilon}_t$ is completely offset by the choice of $c = -1$ already, anything else but $c' = 0$ would add noise back in. The same is true for any higher order polynomial.

3.3 Stochastic process of u_t

The process

$$u_t = \alpha u_{t-1} + \theta\tilde{\epsilon}_{t-1} + \tilde{\epsilon}_t \tag{13}$$

is an ARMA(1,1) process. Using lag operators, (13) can be rewritten as

$$\frac{1-\alpha\mathbb{L}}{1+\theta\mathbb{L}} u_t = \tilde{\epsilon}_t \tag{14}$$

or

$$\begin{aligned}
(1-\alpha\mathbb{L})(1-\theta\mathbb{L}+\theta^2\mathbb{L}^2-\theta^3\mathbb{L}^3+\theta^4\mathbb{L}^4-\dots)\,u_t &= \tilde{\epsilon}_t && (15)\\
(1-\alpha\mathbb{L}-\theta\mathbb{L}+\alpha\theta\mathbb{L}^2+\theta^2\mathbb{L}^2-\alpha\theta^2\mathbb{L}^3-\theta^3\mathbb{L}^3+\dots)\,u_t &= \tilde{\epsilon}_t \\
(1-(\alpha+\theta)\mathbb{L}+(\alpha+\theta)\theta\mathbb{L}^2-(\alpha+\theta)\theta^2\mathbb{L}^3+\dots)\,u_t &= \tilde{\epsilon}_t. && (16)
\end{aligned}$$

Without going through any further derivations, the first step (15) immediately shows that we found an AR(∞) process with

$$u_t = \sum_{s=1}^{\infty} k_s u_{t-s} + \tilde{\epsilon}_t$$

for some k_s. The (not required) further steps show that

$$u_t = (\alpha + \theta) \sum_{s=1}^{\infty} (-\theta)^{s-1} u_{t-s} + \tilde{\epsilon}_t$$

in fact.

3.4 New optimal monetary rule

The optimal monetary rule does not change at all. Both the Central Bank and the Private Sector know all (infinitely many) past realizations of u_{t-s} $(s = 0, 1, 2, \dots)$. So it is still the case that the Private Sector rationally foresees that the expected deviation of the money supply from its mean is equal to the unknown part of the money rule, $m_t^s - \mathbb{E}_{t-1}[m_t^s] = c\tilde{\epsilon}_t + c'(\tilde{\epsilon}_t)^2$. Thus, including any past realizations of u_{t-s} or v_{t-s} is entirely irrelevant. It is still optimal for the central bank to set $c = -1$ and $c' = 0$.

4 Microfoundations of Money Demand

4.1 Interpretation of budget constraint

The individual can choose to allocate her current income Y_t to three different uses: to savings A_{t+1}, or to money holding M_t, or to consumption C_t. The more money M_t she chooses to put aside today (at t), the more of today's income Y_t becomes useful. In the limit $(\frac{M_t}{P_t} \to \infty)$, income is fully useful. In optimum, the individual will choose some intermediate amount of money holdings.

4.2 Choice variables

The individual can choose C_t, A_{t+1}, and M_t. Since she is restricted by the budget constraint, choosing any two of the three implies her choice of the third. Thus, two intertemporal first-order conditions (Euler equations) will be enough to pin the optimal consumption, savings and money holdings paths down.

4.3 First-order conditions

Intertemporal utility $U_t = \sum_{s=t}^{\infty} \beta^{s-t} u(C_s)$[2] can be rewritten as

$$U_t = u(C_t) + \beta U_{t+1} = u(C_t) + \beta u(C_{t+1}) + \beta^2 U_{t+2}. \tag{17}$$

Then, plugging the consumption choice $C_t = (1+r)A_t + \frac{M_{t-1}}{P_t} + g(\frac{M_t}{P_t})Y_t - A_{t+1} - \frac{M_t}{P_t}$ into (17) for t and $t+1$, and maximizing (17) with respect to A_{t+1} and M_t yields the two first-order conditions

$$-u'(C_t) + \beta(1+r)u'(C_{t+1}) = 0 \tag{18}$$

and

$$-\frac{1}{P_t} u'(C_t) \left[1 - g'\left(\frac{M_t}{P_t}\right) \cdot Y_t\right] + \beta \frac{1}{P_{t+1}} u'(C_{t+1}) = 0. \tag{19}$$

Using (18) in (19) yields

$$1 - g'\left(\frac{M_t}{P_t}\right) \cdot Y_t = \frac{1}{1+r} \frac{P_t}{P_{t+1}}. \tag{20}$$

4.4 Money demand under specific functional form for $g(\cdot)$

Given (20) and $Y_s = \overline{Y}$, and using $g(\frac{M_t}{P_t}) = 1 - k\, e^{-\frac{M_t}{P_t}}$ so that $g'(\frac{M_t}{P_t}) = k\, e^{-\frac{M_t}{P_t}}$, we find

$$k\,\overline{Y}\, e^{-\frac{M_t}{P_t}} = 1 - \frac{1}{1+r} \frac{P_t}{P_{t+1}}.$$

Observe that the nominal interest rate must be related to the real interest rate through $1 + i_{t+1} = (1+r)\frac{P_{t+1}}{P_t}$. Then, taking logs of both sides yields

$$\frac{M_t}{P_t} = \ln k\overline{Y} - \ln \frac{i_{t+1}}{1+i_{t+1}}. \tag{21}$$

This function is increasing in income and decreasing in the *nominal* interest rate, just as any *LM*-function should be.

[2] This relationship is correct for $\beta < 1$. If, as in the question, $U_t = \sum_{s=t}^{\infty} \beta^{t-s} u(C_s)$, we must require $\beta > 1$. All the following derivations hold for the more common case where $\beta < 1$ and $U_t = \sum_{s=t}^{\infty} \beta^{s-t} u(C_s)$.