Lecture Notes for February 24, Supplement

8.2 The Shapley-Folkman Theorem

8.2.1 Nonconvex sets and their convex hulls

The convex hull of a set S will be the smallest convex set containing S. The convex hull of S will be denoted con(S). We can define con(S) , for $S \subset R^N$ as follows

$$\text{con(S)} \equiv \{x \mid x = \sum_{i=0}^{N} \alpha^i x^i, \text{where } x^i \in \text{S}, \alpha^i \geq 0 \text{ all i, and } \sum_{i=0}^{N} \alpha^i = 1\}.$$

or equivalently as

$$\text{con(S)} \equiv \bigcap_{S \subset\ T; T\ convex} T.$$

That is con(S) is the smallest convex set in R^N containing S.

8.2.2 The Shapley-Folkman Lemma

Lemma (Shapley-Folkman): Let S^1, S^2, S^3, . . . ,S^m, be nonempty compact subsets of R^N. Let $x \in con(S^1 + S^2 + S^3 + \ldots + S^m)$. Then for each i=1,2,. . . ,m, there is $y^i \in con(S^i)$ so that $\sum_{i=1}^m y^i = x$ and with at most N exceptions, $y^i \in S^i$. Equivalently: Let F be a finite family of nonempty compact sets in R^N and let $y \in con(\sum_{S \in F} S)$. Then there is a partition of F into two disjoint subfamilies F' and F'' with the number of elements in $F' \leq N$ so that $y \in \sum_{S \in F'} con(S) + \sum_{S \in F''} S$.

8.2.3 Measuring Non-Convexity, The Shapley-Folkman Theorem

We now introduce a scalar measure of the size of a non-convexity.

Definition: The radius of a compact set S is defined as
$rad(S) \equiv \inf_{x \in R^N} \sup_{y \in S} |x - y|$.

That is, $rad(S)$ is the radius of the smallest closed ball containing S.

Theorem 8.1 (Shapley - Folkman): Let F be a finite family of compact subsets $S \subset R^N$ and $L > 0$ so that $rad(S) \leq L$ for all $S \in F$. Then for any $x \in con(\sum_{S \in F} S)$ there is $y \in \sum_{S \in F} S$ so that $|x - y| \leq L\sqrt{N}$.

The significance of the Shapley-Folkman theorem is that the sum of a large number of compact non-convex sets is approximately convex. We start with a family of sets F whose elements $S \in F$ are of $rad(S)$, the measure of size, less than or equal to L. The measure of the size of a nonconvexity suggested here is the distance between a point of the convex hull and the nearest point of the underlying set. Adding a few sets together may increase the size of the nonconvexity in the sum; but eventually the radius of the nonconvexity is limited by an upper bound of $L\sqrt{N}$.

8.2.4 Corollary: A tighter bound

Definition: We define the inner radius of $S \subset R^N$ as
$r(S) \equiv \sup_{x \in con(S)} \inf_{T \subset S; x \in con(T)} rad(T)$

Corollary 8.1 Corollary to the Shapley-Folkman Theorem: Let F be a finite family of compact subsets $S \subset R^N$ and $L > 0$ so that $r(S) \leq L$ for all $S \in F$. Then for any $x \in con(\sum_{S \in F} S)$ there is $y \in \sum_{S \in F} S$ so that $|x - y| \leq L\sqrt{N}$.

22.3 A Large Economy without Replication

Recall the Debreu - Scarf proof. The core allocation is shown to be close to competitive equilibrium by showing that the set of preferred net trades is a convex set with the zero vector, 0, on the boundary, and running a supporting hyperplane through 0. Convexity is assured by filling in nonconvexities through replication. Then the normal to the supporting hyperplane, p, is the required competitive equilibrium price vector. The argument without replication follows the same logic, but it cannot fill in the the nonconvexities through replication. Rather, we use the Shapley-Folkman Lemma to show that the nonconvexities are of bounded size, small as a proportion of the number of households as that number becomes large.

The Shapley-Folkman Lemma says that the difference between a sum of sets and the convex hull of the sum is no larger than the N largest summands. In the present argument, we again form the set of preferred net trades and its convex hull. How far is the convex hull of the preferred net trade set from 0? No farther than the N largest summands. Then we can run a supporting hyperplane for this convex hull through a point offset from 0 by the N largest summands. How far is it from supporting the preferred net trade set? No farther than the N largest summands. Thus the normal to the supporting hyperplane supports the core allocation with a discrepancy fixed in size independent of the number of summands. As the economy becomes large, the discrepancy, per head of population, converges to 0.

(C.IV*) (Weak Monotonicity) Let $x, y \in X^i$ and $x >> y$. Then $x \succ_i y$.

We start by measuring the largest of the individual endowments. Define

$$M \equiv \max\{\sum_{i \in S} r_n^i | n = 1, ..., N, S \subseteq H, \#S = N\}$$

Then the N-dimensional vector $(M, M, ..., M)$ is an upper bound on the size of the sum of the endowments of any N-member coalition.

Theorem 22.3: Assume C.IV*, $X^i = \mathbf{R}_+^N$, for all $i \in H$, a pure exchange economy. Let $\{x^{\circ i} | i \in H\}$ be a core allocation for H. Then there is $p \in P$ so that

(i) $\sum_{i \in H} |p \cdot (x^{\circ i} - r^i)| \leq 2M$

(ii) $\sum_{i \in H} |\inf\{p \cdot (x - r^i) | x \succ_i x^{\circ i}\}| \leq 2M$