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Lecture Notes, February 17, 2010

Bargaining and equilibrium: The core of a market economy

Set $X^i = \mathbf{R}_+^N$, all i.

Each $i \in H$ has an endowment $r^i \in \mathbf{R}_+^N$ and a preference quasi-ordering $\succeq_h$ defined on $\mathbf{R}_+^N$.

An allocation is an assignment of $x^i \in \mathbf{R}_+^N$ for each $i \in H$. A typical allocation, $x^i \in \mathbf{R}_+^N$ for each $i \in H$, will be denoted $\{x^i, i \in H\}$. An allocation, $\{x^i, i \in H\}$, is feasible if $\sum_{i \in H} x^i \leq \sum_{i \in H} r^i$, where the inequality holds coordinatewise.

We assume preferences fulfill weak monotonicity (C.IV**), continuity (C.V), and strict convexity (C.VI(SC)).

The core of a pure exchange economy

Definition A ***coalition*** is any subset $S \subseteq H$. Note that every individual comprises a (singleton) coalition.

Definition An allocation $\{x^i, h \in H\}$ is ***blocked*** by $S \subseteq H$ if there is a coalition $S \subseteq H$ and an assignment $\{y^i, i \in S\}$ so that:

(i) $\sum_{i \in S} y^i \leq \sum_{i \in S} r^i$ (where the inequality holds coordinatewise),
(ii) $y^i \succeq_i x^i$, for all $i \in S$, and
(iii) $y^h \succ_h x^h$, for some $h \in S$

Definition The ***core*** of the economy is the set of feasible allocations that are not blocked by any coalition $S \subseteq H$.

- Any allocation in the core must be individually rational. That is, if $\{x^i, i \in H\}$ is a core allocation then we must have $x^i \succeq_h r^i$, for all $i \in H$.
- Any allocation in the core must be Pareto efficient.

(i) The competitive equilibrium is always in the core (Theorem 21.1).

Theorems 22.2 and 22.3 say that

(ii) For a large economy, the set of competitive equilibria and the core are virtually identical. All core allocations are (nearly) competitive equilibria.

The competitive equilibrium allocation is in the core

Definition $p \in \mathbf{R}_+^N$, $p \neq 0$, $x^i \in \mathbf{R}_+^N$, for each $i \in H$, constitutes a competitive equilibrium if

(i) $p \cdot x^i \leq p \cdot r^i$, for each $i \in H$,
(ii) $x^i \succeq_i y$, for all $y \in R_+^N$, such that $p \cdot y \leq p \cdot r^i$, and
(iii) $\sum_{i \in H} x^i \leq \sum_{i \in H} r^i$ (the inequality holds coordinatewise) with $p_k = 0$ for any $k = 1, 2, \ldots, N$ so that the strict inequality holds.

Theorem 21.1 Let the economy fulfill C.II, C.IV**, C.VI(SC) and let $X^i = \mathbf{R}_+^N$. Let p, x^i, $i{\in}H$, be a competitive equilibrium. Then $\{x^i, i \in H\}$ is in the core of the economy.

Proof We will present a proof by contradiction. Suppose the theorem were false. Then there would be a blocking coalition $S{\subseteq}H$ and a blocking assignment $y^i, i \in S$. We have

$\sum_{i \in S} y^i \leq \sum_{i \in S} r^i$(attainability, the inequality holds coordinatewise)
$y^i {\succeq_i} x^i$, for all $i \in S$, *and*
$y^h {\succ_h} x^h$, some $h \in S$.

But x^i is a competitive equilibrium allocation. That is, for all $i \in H$, $p{\cdot}x^i = p{\cdot}r^i$ (recalling Lemma 10.1), and $x^i \succeq_i y$, for all $y \in R_+^N$ such that $p \cdot y \leq p \cdot r^i$.

Note that $\sum_{i \in S} p{\cdot}x^i = \sum_{i \in S} p{\cdot}r^i$. Then for all $i \in S$, $p{\cdot}y^i \geq p{\cdot}r^i$. That is, x^i represents i's most desirable consumption subject to budget constraint. y^i is at least as good under preferences $\succeq_i$ fulfilling C.II, C.IV, C.VI(SC), (local non-satiation). Therefore, y^i must be at least as expensive. Furthermore, for h, we must have $p \cdot y^h > p \cdot r^h$. Therefore, we have

$$\sum_{i \in S} p \cdot y^i > \sum_{i \in S} p \cdot r^i.$$

Note that this is a strict inequality. However, for coalitional feasibility we must have

$$\sum_{i \in S} y^i \leq \sum_{i \in S} r^i.$$

But since $p \geq 0$, $p \neq 0$, we have $\sum_{i \in S} p \cdot y^i \leq \sum_{i \in S} p \cdot r^i$. This is a contradiction. The allocation $\{y^i, i \in S\}$ cannot simultaneously be smaller or equal to the sum of endowments r^i coordinatewise and be more expensive at prices $p, p \geq 0$. The contradiction proves the theorem. QED

Convergence of the core of a large economy

Replication; a large economy

In replication, the economy keeps cloning itself.

duplicate to triplicate, ..., to Q-tuplicate, and so on, the set of core allocations keeps getting smaller, although it always includes the set of competitive equilibria (per Theorem 13.1).

Q-fold replica economy, denoted Q-H. $Q = 1, 2, \ldots$.

$\#H \times Q$ agents.

Q agents with preferences $\succeq_1$ and endowment r^1,

Q agents with preferences $\succeq_2$ and endowment $r^2, \ldots$, and Q agents with preferences $\succeq_{\#H}$ and endowment $r^{\#H}$. Each household $i \in H$ now corresponds to a household type. There are Q individual households of type i in the replica economy Q-H.

Competitive equilibrium prices in the original H economy will be equilibrium prices of the Q-H economy. Household i's competitive equilibrium allocation x^i in the original H economy will be a competitive equilibrium allocation to all type i households in the Q-H replica economy. Agents in the Q-H replica economy will be denoted by their type and a serial number. Thus, the agent denoted i, q will be the qth agent of type i, for each $i \in H, q = 1, 2, \ldots, Q$.

Equal treatment

Theorem 22.1 (Equal treatment in the core) Assume C.IV, C.V, and C.VI(SC). Let $\{x^{i,q}, i \in H, q = 1, \ldots, Q\}$ be in the core of Q-H, the Q-fold replica of economy H. Then for each $i, x^{i,q}$ is the same for all q. That is, $x^{i,q} = x^{i,q'}$ for each $i \in H, q \neq q'$.

Proof of Theorem 14.1 Recall that the core allocation must be feasible. That is,

$$\sum_{i \in H} \sum_{q=1}^{Q} x^{i,q} \leq \sum_{i \in H} \sum_{q=1}^{Q} r^i.$$

Equivalently,

$$\frac{1}{Q} \sum_{i \in H} \sum_{q=1}^{Q} x^{i,q} \leq \sum_{i \in H} r^i.$$

Suppose the theorem to be false. Consider a type i so that $x^{i,q} \neq x^{i,q'}$. For each type i, we can rank the consumptions attributed to type i according to $\succeq_i$.

For each i, let x^{i^*} denote the least preferred of the core allocations to type $i, x^{i,q}, q = 1, \ldots, Q$. For some types i, all individuals of the type will have the same consumption and x^{i^*} will be this expression. For those in which the consumption differs, x^{i^*} will be the least desirable of the consumptions of the type. We now form a coalition consisting of one member of each type: the individual from each type carrying the worst core allocation, x^{i^*}.

Consider the average core allocation to type i, to be denoted $\bar{x}^i$.
$\bar{x}^i = \frac{1}{Q}\sum_{q=1}^{Q} x^{i,q}$.
We have, by strict convexity of preferences (C.VI(SC)),

$$\bar{x}^i = \frac{1}{Q}\sum_{q=1}^{Q} x^{i,q} \succ_i x^{i^*} \text{ for those types } i \text{ so that } x^{i,q} \text{ are not identical,}$$

and

$$x^{i,q} = \bar{x}^i = \frac{1}{Q}\sum_{q=1}^{Q} x^{i,q} \sim_i x^{i^*} \text{ for those types } i \text{ so that } x^{i,q} \text{ are identical.}$$

From feasibility, above, we have that

$$\sum_{i\in H} \bar{x}^i = \sum_{i\in H} \frac{1}{Q}\sum_{q=1}^{Q} x^{i,q} = \frac{1}{Q}\sum_{i\in H}\sum_{q=1}^{Q} x^{i,q} \leq \sum_{i\in H} r^i.$$

In other words, a coalition composed of one of each type (the worst off of each) can achieve the allocation $\bar{x}^i$. However, for each agent in the coalition, $\bar{x}^i \succeq_i x^{i^*}$ for all i and $\bar{x}^i \succ_i x^{i^*}$ for some i. Therefore, the coalition of the worst off individual of each type blocks the allocation $x^{i,q}$. The contradiction proves the theorem. QED

Core$(Q) = \{x^i, i \in H\}$ where $x^{i,q} = x^i, q = 1, 2, \ldots, Q$, and the allocation $x^{i,q}$ is unblocked.

Core convergence in a large economy

As Q grows there are more blocking coalitions, and they are more varied. Any coalition that blocks an allocation in Q-H still blocks the allocation in $(Q+1)$-H, but there are new blocking coalitions and allocations newly blocked in $(Q+1)$-H.

Recall the Bounding Hyperplane Theorem:

Theorem 8.1, Bounding Hyperplane Theorem (Minkowski) Let K be convex, $K \subseteq \mathbf{R}^N$. There is a hyperplane H through z and bounding for K if z is not interior to K. That is, there is $p \in \mathbf{R}^N, p \neq 0$, so that for each $x \in K, p \cdot x \geq p \cdot z$.

Theorem 22.2 (Debreu-Scarf) Assume C.IV**, C.V, C.VI(SC), and let $X^i = \mathbf{R}^N_+$. Let $\{x^{\circ i}, i \in H\} \in \text{core}(Q)$ for all $Q = 1, 2, 3, 4, \ldots$. Then $\{x^{\circ i}, i \in H\}$ is a competitive equilibrium allocation for Q-H, for all Q.

Proof TBA QED