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Why Is There Money?
Endogenous Monetization of an Arrow-Debreu Economy

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Abstract[1].

The "Hahn Problem" is to demonstrate in a Walras-Arrow-Debreu general equilibrium model the positive equilibrium value of fiat money. In a trading post model of N commodities there are $\frac{1}{2}N(N-1)$ commodity-pairwise trading posts. Fiat money's guaranteed value in payment of taxes explains the positive equilibrium price of fiat money. A bid/ask spread at each trading post reflects transaction costs incurred at the post. The large volume of government purchases paid for in fiat money interacts with scale economies in transaction cost to make fiat money the low-transaction-cost commodity. Thus fiat money becomes the unique actively used medium of exchange in general equilibrium. Monetary equilibrium is characterized by all transactions concentrated on the trading posts trading the N commodities for fiat money; the remaining barter trading posts are priced but inactive in equilibrium.

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1 The Hahn Problem

Frank Hahn wrote a variety of fundamental contributions to Walrasian general equilibrium theory and monetary theory. Hahn (1965) articulated the issue that became known as the 'Hahn Problem,' *In a general equilibrium model including money, where money does not enter preferences, can money be shown to have a positive equilibrium value?*[2] He came to a pessimistic interim conclusion, Hahn (1982), "The...challenge that...money poses to the theorist is this: the best developed model of the economy cannot find room for it. The best developed model is, of course, the Arrow-Debreu version of a Walrasian general equilibrium. A first, and...difficult...task is to find an alternative construction without...sacrificing the clarity and logical coherence ... of Arrow-Debreu. "

Why can't the Arrow-Debreu model generate a role for money? Money is a carrier of value between transactions, either briefly as a medium of exchange or over longer periods as an asset. But the Arrow-Debreu model has only one grand transaction for each household and each firm. There is only one grand transaction: all supplies going into a central exchange, all demands coming out, and budgets balancing. With only a single transaction, there is no role for a carrier of value between transactions; there is only one. The approach of this essay is to treat each exchange of one commodity for another as a distinct transaction. With N commodities, there may be $\frac{1}{2}N(N-1)$ distinct transactions. Then there is a function for a carrier (or several carriers) of value between transactions.

Martin Hellwig(1993) restated the Hahn problem as a research agenda, recommending a model of "multiple bilateral exchanges

[2] Paraphrasing Wikipedia. This is the more widely recognized 'Hahn Problem'; the other is instability of a competitive growth path.

in which there is a need for a medium of exchange and the role of money in transactions can be made explicit."

This essay presents a solution to the Hahn problem and a response to Hellwig's agenda. The plan is to provide a general theory that includes a transaction demand for (commodity or fiat) money. Then we formulate an example, following the Walras-Arrow-Debreu model, that derives a positive equilibrium price for government-issued fiat money as the result of elementary price theory. Further, fiat money will be endogenously determined as the universal common medium of exchange. The properties of money are conclusions following from cost and pricing, not assumptions. There is active bilateral exchange and price determination. The positive value and transactions role of fiat money are outcomes of the equilibrium, not assumptions.

Following Hellwig's suggestion, the essential step is to decompose the Walras-Arrow-Debreu market model from a single grand exchange into many smaller transactions, each with its own budget constraint. An endogenous outcome then is that some goods or instruments will be carriers of value moving between transactions. As a result of elementary properties, endogenously determined low transaction cost, fiat money can be derived as the unique carrier of value between exchanges.

Economists have long had explanatons for the curious distinctive properties of fiat money. What is not clear is how to integrate the variety of insights into a Walras-Arrow-Debreu style general equilibrium model. Here are the observations to be included in the Walrasian general equilibrium theory model below:

- The essential point: Why does inherently useless fiat money command a positive price? Typically fiat money is government-issued. Adam Smith (1776) said it; George Knapp (1905[1924]) said it; Abba Lerner (1947) said it. Government makes taxes payable in fiat money. Then fiat money will have a positive price.

- Why does fiat money become the common medium of exchange? That's trickier. The answer is a thick market externality, Rey (2001). High volume markets are liquid markets. Government is a large economic agent spending fiat money heavily in a variety of markets. So fiat money markets (as opposed to markets using alternative currencies or barter exchanges) are high volume markets. They become liquid, with low transaction costs, narrow bid/ask spreads, Menger(1892). A narrow bid/ask spread is liquidity. The high volume instrument becomes the narrow bid/ask spread instrument and is thus priced to be the common medium of exchange.

The approach of this essay is to create a Walrasian general equilibrium model but without the single market for the trade of all commodities. Rather, for each commodity pair there is a pairwise market where the two goods can be traded against each other. There are $\frac{1}{2}N(N-1)$ possible trading pairs. Which pairwise markets are active is part of the outcome of general equilibrium. The equilibrium is *barter* if most goods trade directly against most other goods in equilibrium. The equilibrium is *monetary* if most pairwise markets are priced but inactive, active trade being concentrated on a small band of pairwise markets trading a single good (the medium of exchange) against all other goods. The determination of whether the equilibrium is monetary or barter is part of the equilibrium outcome.

But how is this possible? The Walras-Arrow-Debreu model is famous for denying a role for money! The technical side of the paper brings a variety of generalizations of the Walras-Arrow-Debreu model to bear:

- First, impose the pairwise trade framework with

$\frac{1}{2}N(N-1)$ commodity-pair trading posts. Demonstrate existence of a market clearing equilibrium prices, Starr (2008). Budgets balance at each trading post. Then transactors may need to take the proceeds of trade from one trading post and disburse them on another.

- Second bring in transaction costs generating a gap between bid and ask prices. Duncan Foley (1970) showed us how to treat a Walras-Arrow-Debreu equilibrium with separate buying and selling prices. This treatment is particularly straightforward with linear transaction technology.

- Third, concentration on a single common medium of exchange reflects that it display very low transaction costs. Helene Rey (2001) explains that this results from a thick market externality.

- Existence of general equilibrium can be sustained in the presence of (sufficiently continuous) external effects, Arrow & Hahn (1971).

1.1 Outline of the model

The treatment here presents a trading post model of N commodities with $\frac{1}{2}N(N-1)$ commodity pairwise trading posts. Trading post transaction costs are reflected in the spread between bid and ask prices. Transaction cost technology displays scale economies — high volume markets are low transaction cost markets. Government with taxing power accepts tax payments in the government-issued (otherwise useless) fiat money, ensuring a positive value for fiat money. Government (a large economic agent) spends its fiat money tax receipts at trading posts where fiat money exchanges for goods. High volume at those trading posts generates low transaction costs, narrow bid/ask spreads. But narrow bid/ask spreads are price signals, leading all economic agents to use the fiat money trading posts. Government's taxing power, the large size of government purchases (for fiat money), and scale economies in transaction costs make fiat money the common medium of exchange.

1.2 Liquidity

The most elementary function of money — the medium of exchange — is as a carrier of value held between successive transactions. Carl Menger (1892) reminds us that the distinguishing feature of the medium of exchange should be liquidity. A simple characterization of liquidity is the difference between the bid price and the ask price. A commodity that acts as a medium of exchange is necessarily repeatedly bought (accepted in trade) and sold (delivered in trade). Therefore an instrument with a narrow spread between bid and ask price is priced to encourage households to use it as a carrier of value between trades, as a medium of exchange with relatively low cost.

2 The Trading Post Model

2.1 Trading Posts

The trading post model consists of N commodities traded pairwise at $\frac{1}{2}N(N-1)$ trading posts with distinct bid and ask prices. The bid/ask spread reflects transaction costs. Walras (1874) forms the picture this way (assuming m distinct commodities):

> "we shall imagine that the place which serves as a market for the exchange of all the commodities... for one another is divided into as many sectors as there are pairs of commodities exchanged. We should then have $\frac{\mathrm{m(m-1)}}{2}$

> special markets each identified by a signboard indicating the names of the two commodities exchanged there as well as their ... rates of exchange..."

The trading post model decomposes the trading plans of each household into many separate transactions. The pattern of active trade is endogenously determined as part of the equilibrium of the trading post economy. The general equilibrium of Example 6.2 with a common medium of exchange is developed below. Households create trading plans to optimize utility subject to prevailing prices and subject to a budget constraint at each post. A barter equilibrium occurs when most trading posts are active in equilibrium — most goods trading directly for most other goods. A monetary equilibrium occurs if active trade is concentrated on a few trading posts, those trading the common medium of exchange against most other goods.

Augment the non-monetary Arrow-Debreu model with two additional structures sufficient to give endogenous monetization in equilibrium: multiple budget constraints (one at each transaction, not just on net trade) and transaction costs. The choice of which trading posts a typical household will trade at is part of the household optimization. The equilibrium structure of exchange is the array of trading posts that actually host active trade. The determination of which trading posts are active in equilibrium is endogenous and characterizes the monetary character of trade. The equilibrium is monetary with a unique money if only N trading posts out of $\frac{1}{2}N(N-1)$ are active, those trading all goods against 'money.'

3 Existence of Equilibrium with a Bid/Ask Spread

The model will concentrate on a pure exchange economy with transaction costs. The only resource-using technology is the transaction process embodied in trading firms.

3.1 Commodity Space

There are N elementary commodities. The model distinguishes each commodity k by the alternative commodity against which it may trade, ℓ. A household's commodity trade vector will be $x \in R^{2N(N-1)}$. x is decomposed into two constituent vectors, $x \equiv (x^S, x^B)$. $x^B \in R_+^{N(N-1)}$ represents purchases, and $x^S \in R_-^{N(N-1)}$ represents sales. There are $\frac{1}{2}N(N-1)$ trading posts denoted $\{k, \ell\} \equiv \{\ell, k\}$, the trading post where commodity k is traded for ℓ and vice versa. A single co-ordinate will typically be denoted $x^S(k, \ell)$ or $x^B(k, \ell)$. This is to be read as commodity k at trading post $\{k, \ell\}$ where it is traded for commodity ℓ. There is no entry for $x(k, k)$.

3.2 The General Equilibrium Model with Bid and Ask Prices

The following structure is taken in part from Foley (1970) .

The households and firms face two sets of prices, p^S and p^B in $R_+^{N(N-1)}$. p^S represents wholesale or bid prices. p^B represents retail or ask prices. Let $\Delta =$ unit simplex in $R^{2N(N-1)}$. $p^B \geq p^S$ co-ordinatewise. Following Foley (1970), define $\pi \equiv p^B - p^S$. Then define the price vector $p \equiv (p^S, \pi) \in \Delta$.

3.2.1 Firms

There is a finite population of firms $j \in F$. The firm formulates a transaction plan $(y^{jS}, y^{jB}, w^j) \in R^{3N(N-1)}$. Positive co-ordinates of y^{jB}, y^{jS} indicate sales, and in w indicate inputs to the trading transaction costs. Negative co-ordinates indicate purchases. y^{jS} is the vector of transactions, purchases and sales, the firm makes at bid (wholesale) prices. y^{jB} is the vector of purchases and sales subject to the premium buying (retail) price. Note that in contrast to the households, for the firm, both y^{jS} and y^{jB} can have both positive and negative co-ordinates.

The budget constraint on firm transactions is for each two commodities $k, \ell, = 1, 2, \ldots, N$.

$$p^S(k,\ell) \cdot y^{jS}(k,\ell) + p^B(k,\ell) \cdot y^{jB}(k,\ell)$$
$$+p^B(\ell,k) \cdot y^{jB}(\ell,k) + p^S(\ell,k) \cdot y^{jS}(\ell,k) \geq 0 \quad (\mathcal{B}')$$

Equivalently,

$$p^S(k,\ell) \cdot [y^{jS}(k,\ell) + y^{jB}(k,\ell)] + p^S(\ell,k) \cdot [y^{jB}(\ell,k) + y^{jS}(\ell,k)]$$
$$+\pi(k,\ell) \cdot y^{jB}(k,\ell) + \pi(\ell,k) \cdot y^{jB}(\ell,k) \geq 0 \quad (\mathcal{B}')$$

The concept of profit is not well defined. A suitable maximand for the firm needs to be defined. It is simplest to take the firm's maximand as the conventional specification of profit, $(p^S, p^B) \cdot (y^{jS}, y^{jB}) = (p^S, \pi) \cdot (y^{jS} + y^{jB}, y^{jB})$. The technically possible mix (y^{jS}, y^{jB}, w^j) of purchases, inputs, and sales of firm j is contained in the closed convex set $Y^j \subseteq R^{3N(N-1)}$. Household i owns a proportion Θ^{ij} of firm j, (Foley notation), $1 \geq \Theta^{ij} \geq 0, \sum_{i \in H} \Theta^{ij} = 1$.

Firm j's opportunity set is

$$G^j(p) \equiv Y^j \cap \{(y^{jS}, y^{jB}, w^j) \in R^{3N(N-1)} \text{ fulfills } (\mathcal{B}') \text{ at } p\}$$

Firm j's supply behavior then is

$$S^j(p) \equiv \{(y^{\circ jS}, y^{\circ jB}) = \text{ argmax } [(p^S, p^B) \cdot (y^{jS}, y^{jB})]$$

$$= \text{ argmax } [(p^S, \pi) \cdot (y^{jS} + y^{jB}, y^{jB})]|(y^{jS}, y^{jB}, w^j) \in G^j(p)\}$$

The following assumptions (d.1) to (d.4) on the trading technology are standard in the general equilibrium literature:

(d. 1) $0 \in Y^j$ for all j.(This assumption, together with (c. l), below, assures that there are feasible allocations for the economy.)

(d.2) There is no $(y^{jS}, y^{jB}, w^j) \in Y^j$ with $(y^{jS}, y^{jB}, w^j) \neq 0$ and ≥ 0. (The inequality applies co-ordinatewise. This assumption rules out the possibility of free marketing.)

(d.3) Y^j is a closed convex cone for all j. If $(\bar{y}^{jS}, \bar{y}^{jB}, \bar{w}^j) \in Y^j$ and $(\tilde{y}^{jS}, \tilde{y}^{jB}, \tilde{w}^j) \in Y^j$, then $(\alpha\bar{y}^{jS} + \beta\tilde{y}^j, \alpha\bar{y}^{jB} + \beta\tilde{y}^{jB}, \alpha\bar{w}^j + \beta\tilde{w}^j) \in Y^j$ for $\alpha, \beta \geq 0$.

(d.4)$Y \equiv \sum_j Y^j$ is closed.

(d.1) through (d.4) are in Foley (1970). To keep issues well defined, (d.5) is reassuring

(d.5) $(y^S, y^B, w) \in Y^j$ implies for each $k = 1, 2, \ldots, N$, $\sum_\ell y^S(k, \ell) + \sum_\ell y^B(k, \ell) + \sum_\ell w(k, \ell) \leq 0$.

(d.5) says that firm j must arrange its affairs so that it is (weakly) a net purchaser of each of the N commodities. Deliveries (positive) at one trading post may exceed purchases (negative) there but aggregate purchases by the firm of any commodity must (weakly) exceed sales. This reflects that the economy is pure exchange with some resource expenditure on transaction costs.

3.2.2 The Trading Sector and Attainable Trades

The aggregate trading technology is $Y \equiv \sum_{j\in F} Y^j$. The economy's initial resource vector is $r = \sum_{i\in H} r^i \in R_+^{2N(N-1)}$. Then $(y^S, y^B, w) \in Y$ is said to be *attainable* if , for each $k = 1, 2, \ldots, N$, $\sum_\ell [y^S(k, \ell) + y^B(k, \ell) + w(k, \ell)] \geq -\sum_\ell r(k, \ell)$. $(y^{j\prime S}, y^{j\prime B}, w^{j\prime}) \in$

$Y^{j\prime}$ is said to be *attainable in* $Y^{j\prime}$ if there is $(y^{jS}, y^{jB}, w^j) \in Y^j$ for all $j \in F, j \neq j\prime$, so that $(y^{j\prime S}, y^{j\prime B}, w^{j\prime B}) + \Sigma_{j\in F, j\neq j\prime}(y^{jS}, y^{jB}, w^j)$ is attainable.

Lemma 1 Assume (d.1) through (d.5). Then the set of attainable elements $(y^S, y^B, w) \in Y$ is bounded. And for each $j\prime \in F$, the set of $(y^{j\prime S}, y^{j\prime B}, w^{j\prime}) \in Y^{j\prime}$ attainable in $Y^{j\prime}$ is bounded.

Proof: Starr (2011), Theorem 15.1 .

Choose $c \in R, c > 0$ so that $c > |(y^S, y^B, w)|$ (note the strict inequality) for all attainable $(y^S, y^B, w) \in Y$, and so that $c > |(y^{j\prime S}, y^{j\prime B}, w^{j\prime})|$ (note the strict inequality) for all $(y^{j\prime S}, y^{j\prime B}, w^{j\prime}) \in Y^{j\prime}$ attainable in $Y^{j\prime}$ for all $j\prime \in F$. That is, there is a constant c so that all of the attainable points in any Y^j are strictly contained in a ball of radius c.

Lemma 2 $G^j(p)$ and $S^j(p)$ are homogeneous of degree zero in p.

Let $\tilde{Y}^j \equiv Y^j \cap \{z \in R^{3N(N-1)}, |z| \leq c\}$. Note that $\tilde{Y}^j$ is compact, nonempty.

Firm j's artificially bounded opportunity set is

$$\tilde{G}^j(p) \equiv \tilde{Y}^j \cap \{(y^{jS}, y^{jB}, w^j) \in R^{3N(N-1)} \text{ fulfills } (\mathcal{B}') \text{ at } p\}$$

Firm j's artficiallly bounded supply behavior then is

$$\tilde{S}^j(p) \equiv \{(y^{\circ jS}, y^{\circ jB}) = \text{ argmax } [(p^S, p^B) \cdot (y^{jS}, y^{jB})]$$

$$= \text{ argmax } [(p^S, \pi) \cdot (y^{jS} + y^{jB}, y^{jB})] | (y^{jS}, y^{jB}, w^j) \in \tilde{G}^j(p)\}$$

Let $\tilde{Q}^j(p) \equiv \{(y^{jS}, y^{jB}, w^j) \in \tilde{G}^j(p) | (y^{\circ jS}, y^{\circ jB}) = \text{ argmax } [(p^S, p^B) \cdot (y^{jS}, y^{jB})] = \text{ argmax } [(p^S, \pi) \cdot (y^{jS} + y^{jB}, y^{jB})]\}$

Lemma 3 Assume (d.1) to (d.5). Then $\tilde{G}^j(p)$ is nonempty, convex-valued, continuous (upper and lower hemicontinuous) throughout $p \in \Delta$.

Lemma 4 Assume (d.1) to (d.5). $\tilde{S}^j(p)$ is upper hemicontinuous throughout $p \in \Delta$ and convex-valued. $\tilde{Q}^j(p)$ is upper hemicontinuous throughout $p \in \Delta$ and convex-valued.$\tilde{B}^j(p)$ is upper hemicontinuous throughout $p \in \Delta$ and convex-valued.

3.3 Households

There is a finite set of households H with typical element $i \in H$.

The household i possible consumption set is $W^i \subseteq R^{2N(N-1)}$.

Household i has a share Θ^{ij} of firm j. j makes a distribution to shareholders $[-(y^{jS}+y^{jB}+w^j)] \in R^{2N(N-1)}$ of which i receives $\Theta^{ij}[-(y^{jS}+y^{jB}+w^j)]$ leading to a total of dividend distributions $\sum_{j\in F}\Theta^{ij}[-(y^{jS}+y^{jB}+w^j)]$. Household i has an endowment $r^i \in R_+^{2N(N-1)}$. This treatment has the slightly awkward but harmless suggestion that the household endowment of one commodity is located at the trading post for another.

It is sufficient to characterize household preferences by a well-behaved preference ordering $\succeq_i$ on $R^{2N(N-1)}$. That is the parsimonious way to proceed. The appendix provides a more conventional utility function exposition.

i has a preference ordering $\succeq_i$ on W^i. i makes trades $x^i \in R^{2N(N-1)}$.

$x^i = (x^{iS}, x^{iB})$ reflects $x^{iB} \geq 0, x^{iB} \in R^{N(N-1)}$, the vector of i's purchases, and $x^{iS} \leq 0, x^{iS} \in R^{N(N-1)}$ the vector of i's sales.

The budget constraint on household transactions is for each two commodities $k, \ell, = 1, 2, \ldots, N$.

$$p^S(k,\ell)\cdot x^{iS}(k,\ell) + p^B(k,\ell)\cdot x^{iB}(k,\ell)$$

$$+p^B(\ell,k)\cdot x^{iB}(\ell,k) + p^S(\ell,k)\cdot x^{iS}(\ell,k) \leq 0 \quad (\mathcal{B})$$

Equivalently

$$p^S(k,\ell)\cdot[x^{iS}(k,\ell)+x^{iB}(k,\ell)] + p^S(\ell,k)\cdot[x^{iB}(\ell,k)+x^{iS}(\ell,k)]$$

$$+\pi(k,\ell)\cdot x^{iB}(k,\ell) + \pi(\ell,k)x^{iB}(\ell,k) \leq 0 \quad (\mathcal{B})$$

Let $\mathcal{Y} = (y^{1S}, y^{1B}, w^1, \ldots, y^{jS}, y^{jB}, w^j, \ldots.) \in Y^1 \times Y^2 \times \ldots \times Y^{\#F}$.

The household opportunity set is defined as

$$E^i(p; \mathcal{Y}) \equiv \{(x^{iS}, x^{iB}) | (x^{iS}, x^{iB}) \text{ fulfills } (\mathcal{B}),$$
$$[(x^{iS}, x^{iB}) + \sum_{j \in F} \Theta^{ij}[-(y^{jS} + y^{jB} + w^j)] + r^i] \in W^i\}$$

Household demand behavior is described as

$$D^i(p, \mathcal{Y}) \equiv \{(x^{\circ iS}, x^{\circ iB}) \in E^i(p; \mathcal{Y}) | [(x^{\circ iS}, x^{\circ iB}) + \sum_{j \in F} \Theta^{ij}[-(y^{jS} + y^{jB} + w^j)] + r^i]$$

$$\succeq_i [(x^{iS}, x^{iB}) + \sum_{j \in F} \Theta^{ij}[-(y^{jS} + y^{jB} + w^j)] + r^i] \text{ for all } (x^{iS}, x^{iB}) \in E^i(p; \mathcal{Y})\}$$

Let $\tilde{\mathcal{Y}} \in \tilde{Y}^1 \times \tilde{Y}^2 \times \ldots \times \tilde{Y}^{\#F}$.

The artificially constrained household opportunity set is defined as

$$\tilde{E}^i(p; \tilde{\mathcal{Y}}) \equiv \{(x^{iS}, x^{iB}) | (x^{iS}, x^{iB}) \text{ fulfills } (\mathcal{B}),$$
$$[(x^{iS}, x^{iB}) + \sum_{j \in F} \Theta^{ij}(y^{jS} + y^{jB}) + r^i] \in W^i\} \cap \{z \in R^{N(N-1)}, |z| \leq c\}.$$

Artificially constrained household demand behavior is described as

$$\tilde{D}^i(p; \tilde{\mathcal{Y}}) \equiv \{(x^{\circ iS}, x^{\circ iB}) \in \tilde{E}^i(p; \tilde{\mathcal{Y}}) | [(x^{\circ iS}, x^{\circ iB}) + \sum_{j \in F} \Theta^{ij}[-(y^{jS} + y^{jB} + w^j)] + r^i]$$

$$\succeq_i [(x^{iS}, x^{iB}) + \sum_{j \in F} \Theta^{ij}[-(y^{jS} + y^{jB} + w^j)] + r^i] \text{ for all } (x^{iS}, x^{iB}) \in \tilde{E}^i(p; \tilde{\mathcal{Y}})\}$$

Lemma 5 Let $\tilde{\mathcal{Y}} \in \tilde{Y}^1 \times \tilde{Y}^2 \times \ldots \times \tilde{Y}^{\#F}$. Then $\tilde{E}^i(p; \tilde{\mathcal{Y}})$ and $\tilde{D}^i(p; \tilde{\mathcal{Y}})$ are nonempty homogeneous of degree zero in p. $\tilde{E}^i(p; \tilde{\mathcal{Y}})$ is continuous (upper and lower hemicontinuous) throughout $\Delta \times \tilde{Y}^1 \times \tilde{Y}^2 \times \ldots \times \tilde{Y}^{\#F}$ and convex-valued. $\tilde{D}^i(p; \tilde{\mathcal{Y}})$ is upper hemicontinuous throughout Δ and convex-valued.

3.4 Defining Market Equilibrium

A market equilibrium is a vector of prices (p^{S*}, p^{B*}), a vector $c^{i*} \in W^i$, and $x^{i*} \in R^{2N(N-1)}$ for each household, and a vector $(y^{jS*}, y^{jB*}, w^{j*}) \in Y^j$ for each firm such that

(a)$c^{i*} = r^i + x^{i*} + \sum_{j\in F} \Theta^{ij}[-(y^{*jS} + y^{*jB} + w^{*j})]$ is maximal with respect to $\succeq_i$ in W^i subject to $(\mathcal{B})$ at (p^{S*}, p^{B*});

(b) (y^{jS*}, y^{jB*}) maximizes $(p^{S*}, p^{B*}) \cdot (y^{jS}, y^{jB}) \equiv (p^{S*}, \pi^*) \cdot (y^{jS} + y^{jB}, y^{jB})]$, subject to $(\mathcal{B}')$, and $(y^{jS}, y^{jB}, w^j) \in Y^j$.

(c) $\sum_i(x^{iS*} + x^{iB*}) - \sum_j(y^{jS*} + y^{jB*}) \leq 0$ co-ordinatewise

(d) $p^{S*} \neq 0, p^{B*} \geq p^{S*}, \pi^* \geq 0$ (the inequalities hold co-ordinatewise).

3.4.1 Excess Demand

Let $p = (p^S, \pi) \in \Delta$. Let $(x^{iS}, x^{iB}) \in D^i(p^S, p^B, \mathcal{Y})$ and let $(y^{jS}, y^{jB}) \in S^j(p^S, p^B)$.

Excess demand at $(p^S, \pi) \in \Delta$ and $\mathcal{Y} \in Y^1 \times Y^2 \times \ldots \times Y^{\#F}$ is defined as $Z(p^S, \pi) \equiv \sum_i(x^{iS}, x^{iB}) - \sum_j(y^{jS} + y^{jB})$.

3.5 Sufficient Conditions for Existence of Equilibrium

The following conditions (a) to (d) ensure existence of a quasi-equilibrium; (e) extends to full competitive equilibrium; (f) extends to external effects.

(a.1) The aggregate consumption set $W \equiv \sum_i W^i$ has a lower bound. (This implies that each W^i also has a lower bound.)

(a.2) For each i, W^i is closed and convex.

(b.1) For every attainable consumption $\hat{c}^i$ there exists $c^i \in W^i$ with $c^i \succ_i \hat{c}^i$. (This assumption asserts that the full capacity of the economy is not sufficient to satiate any consumer completely.)

(b.2) For every $c^i \in W^i$ the sets $\{\bar{c}^i \in W^i | \bar{c}^i \succeq_i c^i\}$ and $\{\bar{c}^i \in W^i | \bar{c}^i \preceq_i c^i\}$ are closed in W^i.

(b.3) For every $c^i \in W^i$ the set $\{\bar{c}^i \in W^i | \bar{c}^i \succeq_i c^i\}$ is convex.

(c.1)$0 \in W^i$ for all i.

Conditions (d.1) to (d.5) are stated above.

To move to a full equilibrium, the model with transaction costs needs to ensure positivity of income including accommodating transaction costs. Sufficient conditions follow.

(e.1) $r^i > 0$ for all $i \in H$. The strict inequality holds coordinatewise.

Then we conclude

Lemma 6(Walras's Law): Let $(p^S, \pi) \in \Delta, \pi = p^B - p^S$. Let $(x^{iS}, x^{iB}) \in D^i(p^S, p^B, \mathcal{Y})$ and let $(y^{jS}, y^{jB}) \in S^j(p^S, p^B)$.

Then $p^S \cdot [\sum_i x^{iS} - \sum_j y^{jS}] + p^B \cdot [\sum_i x^{iB} - \sum_j y^{jB}] \leq 0$.

Equivalently, $p^S \cdot [\sum_i x^{iS} - \sum_j y^{jS} + \sum_i x^{iB} - \sum_j y^{jB}] + \pi \cdot [\sum_i x^{iB} - \sum_j y^{jB}] \leq 0$

3.6 Existence of Equilibrium in a Bounded Sub-Economy

Recall the price space is

$\Delta \equiv$ unit simplex in $R^{2N(N-1)}$. And let

$\mathcal{V} \equiv$ a closed ball in $R^{3N(N-1)}$ centered at the origin, strictly including all of the largest attainable net trades of any firm, a ball of radius c.

$\mathcal{V}^{\#F} \equiv$ the $\#F$-fold Cartesian product of $\mathcal{V}$ with itself.

$\mathcal{A} \equiv$ closed ball in $R^{2N(N-1)}$ centered at the origin of radius $(\#F + \#H)c$. This ball strictly includes all of the $\#F$-fold sum of the largest attainable trades of any firm, and the $\#H$-fold sum of the largest attainable trades of any household.

The above sets are all compact and convex. Let Π denote a repeated Cartesian product. The plan is to form a mapping

$\mathcal{T} \equiv \Gamma \times [\Pi_j \tilde{Q}^j] \times \tilde{Z} : \Delta \times \mathcal{V}^{\#F} \times \mathcal{A} \to \Delta \times \mathcal{V}^{\#F} \times \mathcal{A}$. Since the mapping is upper hemicontinuous and convex-valued it will have a fixed point. A fixed point of the mapping will be a market-clearing allocation and prices of the bounded subeconomy. But the bounding constraints are not binding. So the fixed point is also a fixed point of the unbounded counterpart. But then it is a price and equilibrium allocation of the full economy. Here are the correspondences posited above.

Let $\Gamma(z) : \mathcal{A} \rightarrow \Delta$. $z \equiv (z^S, z^B)$. Let $p = (p^S, \pi) \in \Delta$. Then define

$$\Gamma(z) \equiv \{ \text{ argmax}_{(p^S,\pi)\in\Delta}[p^S \cdot (z^S + z^B) + \pi \cdot z^B]\}.$$

Recall $\tilde{S}^j(p) \equiv \{(y^{\circ jS}, y^{\circ jB}) = \text{ argmax } [(p^S, p^B) \cdot (y^{jS}, y^{jB})] = \text{ argmax } [(p^S, \pi) \cdot (y^{jS} + y^{jB}, y^{jB})] | (y^{jS}, y^{jB}, w^j) \in \tilde{Y}^j\}$

Recall $\tilde{Q}^j(p) \equiv \{(y^{\circ jS}, y^{\circ jB}, w^{\circ j}) \in \tilde{G}^j(p) | (y^{\circ jS}, y^{\circ jB}) = \text{ argmax } [(p^S, p^B) \cdot (y^{jS}, y^{jB})] = \text{ argmax } [(p^S, \pi) \cdot (y^{jS} + y^{jB}, y^{jB})]\}$

Recall $\tilde{D}^i(p; \tilde{\mathcal{Y}}) \equiv \{(x^{\circ iS}, x^{\circ iB}) \in \tilde{E}^i(p; \tilde{\mathcal{Y}}) | [(x^{\circ iS}, x^{\circ iB}) + \sum_{j\in F} \Theta^{ij}[-(y^{jS} + y^{jB} + w^j)] + r^i] \succeq_i [(x^{iS}, x^{iB}) + \sum_{j\in F} \Theta^{ij}[-(y^{jS} + y^{jB} + w^j)] + r^i]$ for all $(x^{iS}, x^{iB}) \in \tilde{E}^i(p; \tilde{\mathcal{Y}})\}$

Let $\tilde{\mathcal{Y}} \in \mathcal{V}^{\#F}$, $\tilde{\mathcal{Y}} = \Pi_{j\in F}(y^{jS}, y^{jB}, w^j)$, and let

$\tilde{Z}(p, \tilde{\mathcal{Y}}) \equiv \sum_{i\in H} \tilde{D}^i(p, \tilde{\mathcal{Y}}) - \sum_{j\in F}\{(y^{jS}, y^{jB})\}$. That is, $D^i(p, \tilde{\mathcal{Y}})$ reflects the firm distributions to household i, but the supply side, $\sum_{j\in F}(y^{jS}, y^{jB})$ displays only the marketed portion.

3.7 Equilibrium

Lemma 7 $\Gamma(z) : \mathcal{A} \rightarrow \Delta$ is upper hemi-continuous and convex-valued throughout $\mathcal{A}$.

Lemma 8 Assume (a.1), (a.2), (b.1), (b.2), (b.3), (c.1), (d.1), (d.2), (d.3), (d.4), (e.1), (e.2). Then $\tilde{Q}^j(p) : \Delta \rightarrow \mathcal{V}$, is upper hemicontinuous and convex-valued for all $p \in \Delta$.

Lemma 9 Assume (a.1), (a.2), (b.1), (b.2), (b.3), (c.1), (d.1), (d.2), (d.3), (d.4), (e.1), (e.2). Then $\tilde{Z}(p, \tilde{\mathcal{Y}}) : \Delta \times \mathcal{V}^{\#F} \rightarrow \mathcal{A}$ is upper hemicontinuous and convex-valued throughout $\Delta \times \mathcal{V}^{\#F}$.

Theorem 1: Assume (a.1), (a.2), (b.1), (b.2), (b.3), (c.1), (d.1), (d.2), (d.3), (d.4), (e.1), (e.2). Then the economy has a competitive equilibrium.

Proof: Define the mapping $\mathcal{T} : \Delta \times \mathcal{V}^{\#F} \times \mathcal{A} \to \Delta \times \mathcal{V}^{\#F} \times \mathcal{A}$.

$\mathcal{T}(p, \tilde{\mathcal{Y}}, z) \equiv \Gamma(z) \times [\Pi_j \tilde{Q}^j(p)] \times \tilde{Z}(p, \tilde{\mathcal{Y}})$. $\mathcal{T}$ is upper hemicontinuous and convex-valued throughout $\Delta \times \mathcal{V}^{\#F} \times \mathcal{A}$, so there is a fixed point $(p^*, \tilde{\mathcal{Y}}^*, z^*) \in \Delta \times \mathcal{V}^{\#F} \times \mathcal{A}$. By the Walras's Law, Lemma 6, $p^* \cdot z^* \leq 0$, but $p^* \geq 0$ and p^* is $\text{argmax}_{(p^S,\pi)\in\Delta}[p^S \cdot (z^S + z^B) + \pi \cdot z^B]$ so $z^* \leq 0$. $z^* = \sum_{i\in H} x^{i*} - \sum_{j\in F} y^{j*}$ where $x^{i*} \in \tilde{D}^i(p^*, \tilde{\mathcal{Y}}^*)$ and $y^{j*} \in \tilde{S}^j(p^*)$. But $\sum_{i\in H} x^{i*} \leq \sum_{j\in F} y^{j*}$, so $x^{i*}, i \in H$ is attainable, so $|x^{i*}| < c$. But $|x^{i*}| < c$ and $x^{i*} \in \tilde{D}^i(p^*, \tilde{\mathcal{Y}}^*)$ implies $x^{i*} \in D^i(p^*, \tilde{\mathcal{Y}}^*)$.

3.8 Relation to the Arrow-Debreu Model

The model of this section can include an Arrow-Debreu model as a special case at the cost of relaxing the no free marketing assumption. It is sufficient merely to assume the existence of one firm j' performing costless arbitrage between retail and wholesale markets. That is, technology, $Y^{j'} \subseteq R^{2N(N-1)}$, that buys and sells all dimensions with zero transaction cost. For each k, ℓ, $y^{j'S}(k, \ell) = -y^{j'B}(k, \ell) \in R$. Then all goods can be traded for one another at no transaction cost, and equilibrium prices will be arbitrage-free.

4 EXTERNAL EFFECTS

In this section, the technology Y^j is variable, hence the specification of Y^j enters into the specification of supply. The economic notion here is that transaction technology depends on the thickness of markets. Hence the volume of orders to a trading post helps to determine costs and marginal costs at the post.

(f.1) For each $j \in F, Y^j = \phi^j(\sum_i (x^{iS}, x^{iB}))$.

(f.2) For each $j \in F, \phi^j$ is a continuous (upper and lower hemicontinuous) convex-valued correspondence. $\phi^j : R^{2N(N-1)} \rightarrow R^{3N(N-1)}$.

(f.3) For each $j \in F$, the following set is bounded:

$$\bigcup_{(x^{iS},x^{iB})\in R^{2N(N-1)}} \{(y^{jS}, y^{jB})\text{attainable in } Y^j = \phi^j(\sum_i (x^{iS}, x^{iB}))\}$$

Let $C > 0$ exceed the Euclidean length of any element of the union in (f.3) . The definitions of $\tilde{G}^j(p)$ and $\tilde{Q}^j(p)$ remain essentially unchanged, subsituting C for c and allowing Y^j to be endogenously determined as above. To emphasize the endogeneity, $\tilde{Q}^j(p)$ is rewritten as $\tilde{Q}^j(p, Y^j)$.

Theorem 2: Assume (a.1), (a.2), (b.1), (b.2), (b.3), (c.1), (d.1), (d.2), (d.3), (d.4), (e.1), (e.2), (f.1), (f.2), (f.3). Then the economy has a competitive equilibrium.

Proof:

Let Π indicate multiple Cartesian product.

Recall the price space is

$\Delta \equiv$ unit simplex in $R^{2N(N-1)}$. And let

$\widehat{\mathcal{V}} \equiv$ a closed ball in $R^{3N(N-1)}$ centered at the origin, strictly including all of the largest attainable (y^{jS}, y^{jB}, w^j) of any firm j, a ball of radius C.

$\widehat{\mathcal{V}}^{\#F} \equiv$ the $\#F$-fold Cartesian product of $\widehat{\mathcal{V}}$ with itself.

$\widehat{\mathcal{A}} \equiv$ closed ball in $R^{2N(N-1)}$, centered at the origin, of radius $(\#F + \#H)C$. This ball strictly includes all of the $\#F$-fold sum of the largest attainable trades of any firm, and the $\#H$-fold sum of the largest attainable trades of any household.

$\widehat{\mathcal{A}}^{\#H} \equiv \#H$-fold Cartesian product of $\widehat{\mathcal{A}}$ with itself.

$\mathcal{KC} \equiv \{S \cap \widehat{\mathcal{V}} | S \subseteq R^{3N(N-1)}, S$ (as Y^j) fulfills (d.1) - (d.5) $\}$

$\mathcal{KC}^{\#F} \equiv \#F$-fold Cartesian product of $\mathcal{KC}$ with itself.

Let $p = (p^S, \pi) \in \Delta$ prevailing price vector,

$X^{\#H} \in \Pi_{i\in H}\tilde{D}^i(p,\mathcal{Y}) \subseteq R^{\#H2N(N-1)}$ be the complex of household demands, $X^{\#H} \in \widehat{\mathcal{A}}^{\#H}; X^{\#H} \equiv (x^1, x^2, \ldots, x^{\#H})$

Then define $\tilde{Q}^j(p, Y^j) \equiv \{(y^{\circ jS}, y^{\circ jB}, w^{\circ j}) \in Y^j \cap \widehat{\mathcal{V}} | (y^{\circ jS}, y^{\circ jB}) =$ argmax $[(p^S, p^B)\cdot(y^{jS}, y^{jB}) =$ argmax $[(p^S, \pi)\cdot(y^{jS}+y^{jB}, y^{jB})]$; $(y^{\circ jS}, y^{\circ jB})$ satisfies $\mathcal{B}'\}$,

$\mathcal{Y} \in \Pi_{j\in F}\tilde{Q}^j(p,Y^j) \subseteq R^{\#F3N(N-1)}$ be the complex of firm plans, $\mathcal{Y} = (y^{1S}, y^{1B}, w^1, \ldots, y^{jS}, y^{jB}, w^j, \ldots.) \in \widehat{\mathcal{V}}^{\#F}$

$Y^{Agg} \in \Pi_{j\in F}\phi^j(\sum_{i\in H} x^i) \subseteq R^{\#F3N(N-1)}$ be the complex of (endogenously determined) transaction technologies, $Y^{Agg} \in \mathcal{KC}^{\#F}$; $Y^{Agg} \equiv (Y^1, Y^2, \ldots, Y^{\#F})$.

$\tilde{Z}(p,\mathcal{Y}) \equiv \sum_{i\in H}\tilde{D}^i(p,\mathcal{Y}) - \{\sum_{j\in F}(y^{jS}, y^{jB})\}$

$z \in \tilde{Z}(p,\mathcal{Y}) \subseteq R^{2N(N-1)}$ be the vector of excess demands, $z = \sum_{i\in H} x^i - \sum_{j\in F} y^j$

$\Gamma(z)$ be the price adjustment correspondence.

$$\Gamma(z) \equiv \{\ \text{argmax}_{(p^S,\pi)\in\Delta}[p^S\cdot(z^S+z^B)+\pi\cdot z^B]\}.$$

Let

$(p, X^{\#H}, \mathcal{Y}, Y^{Agg}, z) \in \Delta \times \widehat{\mathcal{A}}^{\#H} \times \widehat{\mathcal{V}}^{\#F} \times \mathcal{KC}^{\#F} \times \widehat{\mathcal{A}}$.

Let $\widehat{\mathcal{T}} : \Delta \times \widehat{\mathcal{A}}^{\#H} \times \widehat{\mathcal{V}}^{\#F} \times \mathcal{KC}^{\#F} \times \widehat{\mathcal{A}} \to \Delta \times \widehat{\mathcal{A}}^{\#H} \times \widehat{\mathcal{V}}^{\#F} \times \mathcal{KC}^{\#F} \times \widehat{\mathcal{A}}$

$\equiv \Gamma(z) \times \Pi_{i\in H}\tilde{D}^i(p,\mathcal{Y}) \times \Pi_{j\in F}\tilde{Q}^j(p,Y^j) \times \Pi_{j\in F}\phi^j(\sum_{i\in H} x^i) \times \tilde{Z}(p,\mathcal{Y})$

Note that $\Gamma, \tilde{D}^i, \tilde{Q}^j, \phi^j,$ and $\tilde{Z}(p,\mathcal{Y})$ are all well defined, upper hemicontinuous, and convex-valued throughout $\Delta \times \widehat{\mathcal{A}}^{\#H} \times \widehat{\mathcal{V}}^{\#F} \times \mathcal{KC}^{\#F} \times \widehat{\mathcal{A}}$. Then the proof will apply the Kakutani Fixed Point Theorem, to generate a fixed point, $(p^\circ, x^{\#H\circ}, \mathcal{Y}^\circ, Y^{Agg\circ}, z^\circ)$. The proof will argue that the fixed point is a market-clearing equilibrium.

Note compactness of the attainable set, convexity, continuity.

By the Walras's Law, Lemma 6, $p^\circ \cdot z^\circ \leq 0$, but $p^\circ \geq 0$ and p° is

$\text{argmax}_{(p^S,\pi)\in\Delta}[p^S\cdot(z^S+z^B)+\pi\cdot z^B]$ so $z^\circ \leq 0$. $z^\circ = \sum_{i\in H} x^{i\circ} - \sum_{j\in F} y^{j\circ}$ where $x^{i\circ} \in \tilde{D}^i(p^\circ, \tilde{\mathcal{Y}}^\circ)$ and $y^{j\circ} \in \tilde{S}^j(p^\circ)$. But $\sum_{i\in H} x^{i\circ} \leq \sum_{j\in F} y^{j\circ}$, so $x^{i\circ}, i \in H$ is attainable, so $|x^{i\circ}| < C$. But $|x^{i\circ}| < C$ and $x^{i\circ} \in \tilde{D}^i(p^\circ, \tilde{\mathcal{Y}}^\circ)$ implies that the length constraint to C is not binding, so $x^{i\circ} \in D^i(p^\circ, \tilde{\mathcal{Y}}^\circ)$.

Hence markets clear and the households and firms are optimizing subject to budget and technology (but not length) constraints. The price and allocation is a general equilibrium.

5 MEDIUM OF EXCHANGE

In a competitive general equilibrium, let $x^{iS}(k,\ell) < 0, x^{iB}(k,m) > 0$, for some ℓ, m. Then k is a medium of exchange. How can we distinguish between k's role as a medium of exchange and simple arbitrage? There will be no arbitrage in a general equilibrium. Why? Any profitable arbitrage will be infinitely profitable at infinite scale. Hence it cannot occur in equilibrium. Remaining transactions, characterized by buying and selling the same commodity by the same household, mean that the commodity is being used as a medium of exchange.

6 Example: Equilibrium with a Thick Market Externality and a Unique Medium of Exchange

6.1 Households

Let $N \geq 3$. Ω denotes the greatest integer $\leq (N-1)/2$. Begin with a population of $[10 \times N \times \Omega]$ households. Let the households $i \in H$ be enumerated in the following way:

$a.m.n$ where

$a = 1, 2, \ldots, N$,

$m = 1, 2, ..., 10,$
$n = (a+1)(\text{mod } N), (a+2)(\text{mod } N), ..., (a+\Omega+1)(\text{mod } N)$

The typical household $a.m.n$ is endowed with good a, in quantity A, prefers good n, and there are 10 identically situated households denoted by m. The reason for introducing 10 identical individuals is to provide the flexibility to add a smaller number of differing individuals. They will find that the thick markets externality generated by the larger number leads them to use the same medium of exchange as the large number. The notion is that this will be true even if there is a double coincidence of wants for the small differing number.

Household $a.m.n$'s utility function is

$$u^{a.m.n}(x) = x_n \tag{1}$$

That is, household $a.m.n$ values good n only and gladly trades his endowed good a for n.

This represents a population of households displaying a complete absence of double coincidence of wants. Assume that there are 10Ω households endowed with each good and each household desires a good different from its endowment. There are 10Ω households endowed with good 1, preferring respectively, goods 2, 3, 4, ..., $\Omega + 1$. There are 10Ω households endowed with good 2, preferring respectively goods 3, 4, 5, ..., $\Omega + 2$. The roll call of households proceeds so forth.

One way to think of the population is that its elements are set round a clock-face at a position corresponding to the endowed good, a, eager to acquire n. n being 1, 2, ... , Ω, steps clockwise from a. Absent "double coincidence of wants", Jevons (1875), there is unlikely to be successful barter. In this example, for each household endowed with good a and desiring good n, there is no precise mirror image endowed with n desiring a. Nevertheless, there are 10Ω households endowed with A of commodity 1, and 10Ω households strongly preferring commodity 1 to all others. That is true for each good. Thus gross supplies equal gross demands, though there is no immediate opportunity for any two households to make a mutually advantageous trade. Jevons (1875) tells us that this is precisely the setting where money is suitable to facilitate trade.

6.2 Transaction Costs and Monetary Equilibrium

The fraction $\frac{\sum_i x^{iS}(k,\ell)}{10\times\Omega\times A}$ represents the fraction of total possible household offers of commodity k taking place at trading post $\{k,\ell\}$. We'd like a means to represent transaction efficiency increasing in this value. The example developed here uses a simple specification of transaction cost in terms of the transaction technology Y^j. The underlying economics is that there is an external effect. The offer volume fraction affects the efficiency of the transaction technology. A high volume of trading offers reduces unit transaction costs.

Consider pure trading technology, Y^j, with a 'iceberg' style transaction cost. The firm buys wholesale, sells retail, and wastes a lot of the goods in transit. There is an external economy. High offer volume in k at trading post $\{k,\ell\}$ reduces transaction costs on trading k at the post.

At low offer volume in k at $\{k,\ell\}$, a typical trading firm buys wholesale twice as much as it sells retail. Transaction costs are equal to selling volume. At maximum volume a trading firm's buying and selling quantities are nearly the same, transaction costs have been reduced nearly to zero. The example would be even simpler if the costs were reduced to zero, but that would violate the 'no free marketing' assumption.

At near-zero offer volume, $0 < y^{jB}(k,\ell) = -\frac{1}{2}y^{jS}(k,\ell)$. More generally, $0 < y^{jB}(k,\ell) = -\frac{1}{\delta}y^{jS}(k,\ell)$ where $\delta = 1+\gamma$, $\gamma = \max[1 - \frac{\sum_i x^{iS}(k,\ell)}{\rho \sum_{i\in H}\sum_{m=1,\ldots,N} r^i(k,m)}, 0] + 0.001$. The interpretation here is that γ represents the premium over one-for-one trade needed to cover transaction cost. ρ is a proportion of the total amount of good k in the economy so that if the proportion ρ is offered through the trading post $\{k,\ell\}$ then the trading volume is sufficient to bring the premium $\pi(k,\ell)$, the transaction cost, down to near zero.

Claim: There is an equilibrium in an economy with this technology. Further, there is an equilibrium with transactions concentrated on a single intermediary commodity.

EXAMPLE 6.2: Let the population and transaction technology be as above. Let $\rho = 1$. Choose $\tilde{\mu} \in \{1,2,\ldots,N\}$. Set $1 = p^S(\tilde{\mu},\ell), 1.001 = p^B(\tilde{\mu},\ell)$, for all $\ell = 1,2,\ldots,N$. For all $k \in \{1,2,\ldots,N\}, k \neq \tilde{\mu}$, set $1 = p^S(k,\ell), 2 = p^B(k,\ell)$, for all $\ell = 1,2,\ldots,N$. Let $p^B(k,\tilde{\mu}) = 1.001, p^S(k,\tilde{\mu}) = 1$. Set $x^{a.m.nS}(a,\tilde{\mu}) = -A, x^{a.m.nB}(\tilde{\mu},a) = 0.999A$. $x^{a.m.nS}(\tilde{\mu},n) = -0.999A$, $x^{a.m.nB}(n,\tilde{\mu}) = 0.998A$. Denote a typical firm $\tilde{j}$ as a market maker in the trading posts $\{\tilde{\mu},k\}, k = 1,2,\ldots,N$.

Let $\sum_{\tilde{j}} y^{\tilde{j}S}(\tilde{\mu},k) = -10\times\Omega\times A = \sum_{\tilde{j}} y^{\tilde{j}S}(k,\tilde{\mu})$; $\sum_{\tilde{j}} y^{\tilde{j}B}(k,\tilde{\mu}) = 9.98\times\Omega\times A$; $\sum_{\tilde{j}} y^{\tilde{j}B}(\tilde{\mu},k) = 9.99\times\Omega\times A$. At this trading volume, with the specified external economy, $\tilde{j}$ breaks even with $\pi(\tilde{\mu},k) = \pi(k,\tilde{\mu}) = 0.001$.

6.3 Transactions fulfilling double coincidence of wants could be implemented by barter. But they are traded using the high volume medium of exchange.

Consider a small subpopulation with endowment/demand so that they display a double coincidence of wants. Each has and will willingly supply what the other wants. Jevons (1875), Smith (1776), and more recently Kiyotaki and Wright (1989, 1993) suggest that their exchange may take place without the use of a medium of exchange. In actual practice direct barter is even rarer than the double coincidence of wants. Grocery store employees buy their food products for money, not by direct exchange of labor for goods. University of California faculty whose children are enrolled at Berkeley pay tuition in money, not by direct exchange of lectures for enrollment.

Why use money when direct barter will do? The model of this section suggests an answer. Barter markets are thin. Thin markets have high transaction costs. Concentrating trade on thick markets, even if it requires some indirect trade, can achieve lower transaction costs.

EXAMPLE : Start from the previous example. Choose $N \geq 5$. Augment the population by two households: 1.1.Ω+2, Ω+2.1.1. Each hopes to supply what the other wants. In trade through $\tilde{\mu}$, $p^S(1, \tilde{\mu}) = p^S(\tilde{\mu}, 1) = p^S(\Omega + 2, \tilde{\mu}) = p^S(\tilde{\mu}, \Omega + 2) = 1, p^B(1, \tilde{\mu}) = p^B(\tilde{\mu}, 1) = p^B(\Omega + 2, \tilde{\mu}) = p^B(\tilde{\mu}, \Omega + 2) = 1.001$. In direct trade $p^S(1, \Omega + 2) = p^S(\Omega + 2, 1) = 1$. However, $p^B(1, \Omega + 2) = p^B(\Omega + 2, 1) = 2$. Thus $\pi(1, \Omega + 2) = \pi(\Omega + 2, 1) = 1$. Bottom line: Trading $\Omega + 2$ directly for 1 means that half of the goods are lost in transit. Trading indirectly throught $\tilde{\mu}$, means that most of the goods go to the principal traders with only $\frac{2}{10}$ of 1% expended in transaction costs.

6.4 Fiat Money

In order to study fiat money we introduce agent G, government, with the unique power to issue fiat money. Fiat money is intrinsically worthless; it enters no one's utility function. But government, agent G, is uniquely capable of declaring it acceptable in payment of taxes. Adam Smith (1776), Georg Knapp(1905, [1924]), and Abba Lerner (1947) remind us of taxation's role.

Government, agent G, sells tax receipts, the $N+1^{\mathrm{st}}$ good. It also sells good $N+2$, an intrinsically worthless instrument, (latent) fiat money, that agent G undertakes to accept in payment of taxes, that is, in exchange for $N+1$.

Government, agent G, uses its revenue to purchase a variety of goods $n = 1, ..., N$, in the amount x_n^G.

Good $N+2$ represents latent fiat money. Government, G, sells $N+1$ (tax receipts) for $N+2$ at a fixed ratio of one-for-one. The trading post $\{N+1, N+2\}$ where tax receipts are traded for $N+2$ operates with minimal transaction cost, one tenth of 1%. The market clearing price of $N+2$ is to be determined in equilibrium. Its acceptability in payment of taxes will encourage positive value. If, in addition, $N+2$ trades at sufficiently low transaction cost, then it becomes the common medium of exchange.

EXAMPLE 6.4 In this example, we build on the Example 6.2. In that example a commonly traded good produced a thick market externality giving that good a low transaction cost and securing its position as the common medium of exchange. Here, we'll take the latent fiat money instrument, $N+2$, and note two properties it may have. It is acceptable in payment of taxes, and so becomes valuable. Agent G, government, expends $N+2$ for government purchases. G is a large economic agent. Its high volume of trade generates the thick market externality. Then the trick is done; $N+2$ has low transaction costs and becomes the common medium of exchange.

To avoid transparently forcing the result that fiat money be the sole common medium of exchange, it is useful to include the possibility of paying taxes in kind. Assume that the transaction technology described above works for the trade of k for $N+1$, $k = 1, 2, \ldots, N$. Set $1 = p^S(k, \ell), 2 = p^B(k, \ell)$, for all $k, \ell =$

$1, 2, \ldots, N, N+1; k \neq \ell$. This expression says that good k can be traded directly for good ℓ including $\ell = N+1$, tax receipts. But it is priced for low volume. So transaction costs are high, $\pi(k, \ell) = \pi(\ell, k) = 1 = \pi(N+1, k)$.

Start from example 6.2. Let $\tilde{\mu} = N+2$. Let each household $a.m.n$ have a designated tax bill $\tau^{a.m.n} = \tau^{\circ} = \rho A > 0$. That is, taxes constitute the proportion ρ of endowment, payable in $k = 1, 2, \ldots, N, N+2$. Recall that ρ is the proportion of total trade in a typical commodity that generates maximal external effect, giving low transaction costs.

Let household $a.m.n$'s maximand be the utility function $u^{a.m.n}(x) = x_n - 2[\max[(\tau^{a.m.n} - x^{a.m.n}_{N+1}), 0]]$. Thus, households are eager to pay taxes, up to the tax bill, and get no utility from tax payments beyond the tax bill.

Set $1 = p^S(N+2, \ell), 1.001 = p^B(N+2, \ell)$, for all $\ell = 1, 2, \ldots, N$. $p^B(\ell, N+2) = 1.001, p^S(\ell, N+2) = 1$.

$p^S(N+1, N+2) = 1, p^B(N+1, N+2) = 1.001$. Let $x^G(n, N+2) = 9.99N\Omega\tau^{\circ}$ and $x^G(N+2, n) = -10N\Omega\tau^{\circ}$.

$x^G(N+2, N+1) = 10.01N\Omega\tau^{\circ}$. $x^G(N+1, N+2) = -10N\Omega\tau^{\circ}$. For all $a.m.n$,

$x^{a.m.nS}(N+2, n) = -.999(A - \tau^{\circ}), x^{a.m.nB}(n, N+2) = 0.998 \cdot (A - \tau^{\circ})$.

$x^{a.m.nS}(N+2, N+1) = -1.001\tau^{\circ}, x^{a.m.nB}(N+1, N+2) = \tau^{\circ}$.

set $x^{a.m.nB}(N+2, a) = .999A, x^{a.m.nS}(a, N+2) = -A$. Then the equilibrium of Example 6.2 is sustained. In this example, $N+2$ fiat money, becomes the sole common medium of exchange.

But that equilibrium need not be unique. The scale economy, as in Example 6.2, tends to a single common medium of exchange in equilibrium. But any commodity k could function as the common medium of exchange, $\tilde{\mu}$. If agent G will accept k in payment of taxes, then with high trading volume, trade in k, including for tax payments, carries low transaction cost.

6.5 Converging to a Fiat Money Equilibrium

6.5.1 Tatonnement adjustment process for fiat money equilibrium

In a decentralized competitive market, will the economy converge on a single common medium of exchange? The scale economy in transaction cost posited above means that there may be a natural monopoly in the medium of exchange. The example below suggests that a dynamic adjustment will converge on that monopoly arrangement.

Prices will be adjusted by a marginal cost pricing auctioneer. Specify the following adjustment process for prices.

> STEP 0: Price at low trading volume.
>
> CYCLE 1
>
> STEP 1: Households compute their desired trades at the posted prices and report them for each pairwise market.
>
> STEP 2: Marginal cost prices are computed for each pairwise market based on the outcome of STEP 1. Prices are adjusted upward for goods in excess demand at a trading post, downward for goods in excess supply, with the bid-ask spead adjusted to marginal transaction cost. Note that linear costs are posited.
>
> CYCLE 2 Repeat STEP 1 (at the new posted prices) and STEP 2.
>
> CYCLE 3, CYCLE 4, ... repeat until the process converges and trading posts clear.

The plausible adjustment process above explains why government-issued fiat money becomes the unique common medium of exchange —- and would do so even in the absence of legal tender rules. Government has two distinctive characteristics: it has the power to support the value of fiat money by making it acceptable in payment of taxes; it is a large economic presence undertaking a high volume of transactions in the economy. Hence, government can make its fiat money the common medium of exchange merely by using it in government transactions. The scale economies implied will make fiat money the low transaction cost instrument and hence the most suitable medium of exchange, not just for government but for all transactors.

6.5.2 Convergence

EXAMPLE 6.5 Sufficiently high taxes generate scale economy with narrow bid ask spread. Leading to monetary equilibrium with commodity $N+2$ as the unique medium of exchange.

As in Example 6.4, let each household $a.m.n$ have a designated tax bill $\tau^{a.m.n} = \tau^{\circ} = \rho A, \quad \rho > 0$. Recall that ρ represents the proportion of endowment that, when offered in trade, generates maximum scale economy in transaction cost. Taxes constitute the proportion ρ of endowment, payable in $N+2$. Let $u^{a.m.n}(x) = x_n - 2[\max[(\tau^{a.m.n} - x^{a.m.n}_{N+1}), 0]]$. Household utility functions represent that households are eager to pay their taxes, and pay no more. Then the adjustment process converges to the equilibrium of Example 6.4.

STEP 0: $p^B(k,\ell) = 2, p^S(k,\ell) = 1; k,\ell, = 1,\ldots,N$. $p^B(N+1, N+2) = 1.001, 1 = p^S(N+1, N+2)$. $p^B(N+2,k) = 2, p^S(N+2,k) = 1; k = 1,\ldots,N$. $p^B(N+1,k) = 2, p^S(N+1,k) = 1, k, = 1,\ldots,N$.

CYCLE 1, STEP 1: $x^{a.m.nB}(N+2,a) = \tau^{\circ}$, $x^{a.m.nS}(a,N+2) = -2\tau^{\circ}$. $x^{a.m.nB}(n,a) = 0.5(A-2\tau^{\circ}), x^{a.m.nS}(a,n) = -(A-2\tau^{\circ})$. $x^{GB}(a,N+2) = 10\tau^{\circ}N$, $x^{GB}(N+2,a) = -5\tau^{\circ}N$.

CYCLE 1, STEP 2: $p^B(k,\ell) = 2, p^S(k,\ell) = 1; k,\ell, = 1,\ldots,N$. $p^B(N+1,N+2) = 1.001, 1 = p^S(N+1,N+2)$. $p^B(N+2,k) = 1.001, p^S(N+2,k) = 1; k = 1,\ldots,N$. $p^B(N+1,k) = 2, p^S(N+

$1, k) = 1, k, = 1, \ldots, N$.

CYCLE 2, STEP 1: $x^{a.m.nB}(N+2, a) = \tau^\circ$, $x^{a.m.nS}(a, N+2) = -\tau^\circ$. $x^{a.m.nB}(n, a) = 0, x^{a.m.nS}(a, n) = 0$. $x^{GB}(a, N+2) = 10\tau^\circ N$, $x^{GB}(N+2, a) = -10\tau^\circ N$. $x^{a.m.nB}(n, N+2) = 0.999 \cdot (A - \tau^\circ), x^{a.m.nS}(a, N+2) = -A. x^{a.m.nB}(N+1, N+2) = \tau^\circ, x^{a.m.nB}(N+2, N+1) = -\tau^\circ$.

CYCLE 2, STEP 2: $p^B(k, \ell) = 2, p^S(k, \ell) = 1; k, \ell, = 1, \ldots, N$. $p^B(N+1, N+2) = 1 = p^S(N+1, N+2)$. $p^B(N+2, k) = 1.001, p^S(N+2, k) = 1; k = 1, \ldots, N$. $p^B(N+1, k) = 3, p^S(N+1, k) = 1, k, = 1, \ldots, N$. **Unchanged from CYCLE 1. Convergence.**

CYCLE 3, STEP 1: **Unchanged from CYCLE 2. Convergence.**

What's happening in this example? Scale economies are taking their course! Government expenditures in all goods markets in exchange for $N+2$ (and large household demand to acquire $N+2$ to finance tax payments) result in a large trading volume on the trading posts for good $N+2$ versus $n = 1, ..., N$. Volume is large enough that scale economies kick in. The marginal cost pricing auctioneer adjusts prices, the bid/ask spread, to reflect the scale economies. The bid/ask spreads incurred on trading k for ℓ by way of good $N+2$ become considerably narrower than on trading k for ℓ directly. The price system then directs each household to the market $\{k, N+2\}$ where its endowment is traded against good $N+2$. The household sells all its endowment there for $N+2$ and trades $N+2$ subsequently for tax payments and desired consumption. Scale economy has turned $N+2$ from a mere tax payment coupon into 'money,' the unique universally used common medium of exchange.

7 Summary

This paper has focused on two principal issues. First is to demonstrate existence of general equilibrium in a trading post model with several distinct characteristics:

- transaction costs leading to a bid/ask spread in prices,
- an externality determining transaction technology,
- separate budget constraint at each trading post,
- an endogenous medium of exchange function.

Of course, this result requires continuity and convexity everywhere, except that it admits a scale economy in transaction costs external to the individual firms.

The remaining result is to display an example where useless fiat money has a positive equilibrium price and is endogenously determined to be the common medium of exchange. Money as the medium of exchange, and fiat money with a positive equilibrium price are presented in an elaboration of the Arrow-Debreu model. Hahn's problem has a solution.

Appendix: Conventional Utility Function; A Digression

It takes a bit of imagination to conceive of household preferences over transactions at a vast array of trading posts. Of course the household's preferences on transactions are based on underlying preferences on consumptions. That's the notion of this section. One interpretation of preferences on $2N(N-1)$ distinct commodities, in more conventional language, would be as follows. Household i has a utility function $u^i : R_+^N \to R$, and grand utility function $\mathcal{U}^i(x) : R_+^{2N(N-1)} \to R$. Let

$$Q_k(x) \equiv \sum_{\ell=1}^{N} [x^{iS}(k,\ell)+x^{iB}(k,\ell)], \quad Q(x) \equiv (Q_1(x), Q_2(x), \ldots, Q_N(x)),$$

and

$$\mathcal{U}^i(x) \equiv u^i(Q(x)).$$

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