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Approximation of Points of the Convex Hull of a Sum of Sets by Points of the Sum: An Elementary Approach*

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The Shapley–Folkman theorem and its corollaries [1, 2, 3, 4, 5, 6, 8] provide strong bounds on the distance between the sum of a family of non-convex sets and the convex hull of the sum. Proofs of the theorem are non-constructive, and require moderately advanced analysis. The proof developed below is based on elementary considerations. It provides an approximation sequentially with the successive addition of sets to the sum. The approximation is not so close as that provided by the Shapley–Folkman theorem, but for any given point of the convex hull we will find a specific point in the sum within a previously determined bound on the distance between the two.

Particular virtues of the bounds associated with the Shapley–Folkman theorem are the relatively tight approximation developed and its behavior as the number of summands becomes large. The bounded distance between the sum and its convex hull depends not on the number of sets summed (denoted m) as this number becomes large, but rather on the dimensionality of the space (denoted N). Thus as the number of summands becomes large, the average discrepancy between the sum and its convex hull converges to 0 as $1/m$. The bounds developed below, on the contrary, vary as $m^{1/2}$ so that the average discrepancy converges to 0 as $1/m^{1/2}$. The virtue of the results here is their comparative ease of proof and the sequential construction making it relatively easy to find a point of the sum nearby to any chosen point of the convex hull.

For $S \subset R^N$, S compact, we define several measures of size and non-convexity. con S denotes the convex hull of S.

* A sketch of the analysis of this paper was developed in 1965 in an early draft of [8]. It was not published then, since it seemed to be superseded by the Shapley–Folkman theorem. Subsequent discussions have convinced me of its independent interest, particularly in relation to theorem 1 of [4] and the theorem of [7].

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The *radius* of S is

$$\operatorname{rad}(S) \equiv \inf_{x \in R^N} \sup_{y \in S} |x - y|.$$

$\operatorname{rad}(S)$ is the radius of the smallest sphere that can fully contain S.

The *inner radius* of S is

$$r(S) = \sup_{x \in \operatorname{con} S} \inf_{\substack{T \subset S \\ T \text{ spans } x}} \operatorname{rad}(T).$$

$r(S)$ is the smallest radius of a ball centered in $\operatorname{con} S$ so that the ball is certain to contain points of S that span its center. $r(S)$ is a measure of the size of the nonconvexities (holes) in S.

The *inner diameter* of S, $d(S)$ is simply twice the inner radius;

$$d(S) = 2r(S).$$

Note that a sphere centered at any point, y, of $\operatorname{con} S$ with radius $d(S)$ contains a set of points spanning y. We make use of this property to prove

LEMMA. *Let $U \subset R^N$, U compact, $v \in R^N$, $y \in \operatorname{con} U$. Then there is $x \in U$ such that $|x - y| \leqslant d(U)$ and $v \cdot (x - y) \leqslant 0$.*

Proof. There are $x^j \in U$, $j = 1, \ldots,$ (at most $N + 1$ points x^j are required) so that for some $a^j \geqslant 0$, $\Sigma a^j = 1$, $\Sigma a^j x^j = y$ and $|x^j - y| \leqslant d(U)$. We then have

$$\Sigma a^j (x^j - y) = 0,$$

$$v \cdot \Sigma a^j (x^j - y) = 0,$$

$$\Sigma a^j v \cdot (x^j - y) = 0,$$

so for some j, $v \cdot (x^j - y) \leqslant 0$. Let this x^j be the required x. Q.E.D.

Let S^i be a countable collection of compact subsets of R^N. Further, let $d(S^i) \leqslant D$ for all i. Describe y as $\sum_{i=1}^m y^i$, where $y^i \in \operatorname{con} S^i$. Choose x^1 in S^1 so that $|y^1 - x^1| \leqslant D$. Find $x^2 \in S^2$ so that $(y^1 - x^1) \cdot (y^2 - x^2) \leqslant 0$ and $|y^2 - x^2| \leqslant D$. The lemma assures us that there is such x^2. Since $(y^2 - x^2)$ meets $(y^1 - x^1)$ at an acute angle, $|\sum_{i=1}^2 y^i - \sum_{i=1}^2 x^i|^2 \leqslant \sum_{i=1}^2 |y^i - x^i|^2$. This is argued more completely in the proof of the Proposition. Hence $|\sum_{i=1}^2 y^i - \sum_{i=1}^2 x^i| \leqslant D(2)^{1/2}$. Proceeding sequentially, x^k is chosen in S^k within a distance D of y^k and at an acute angle to $(\sum_{i=1}^{k-1} y^i - \sum_{i=1}^{k-1} x^i)$. The lemma assures that this is possible. This gives $|\sum_{i=1}^k y^i - \sum_{i=1}^k x^i| \leqslant D(k)^{1/2}$.

Setting $k=m$, $|\sum_{i=1}^m y^i - \sum_{i=1}^m x^i| \leqslant D(m)^{1/2}$. $\sum_{i=1}^m x^i$ is the desired point in $\sum_{i=1}^m S^i$ relatively near to y. Further, as the number of sets summed, m, increases, the average discrepancy of the approximation $D(m)^{1/2}/m$ converges to 0 as $1/(m)^{1/2}$.

PROPOSITION. *Let $x \in \operatorname{con} \sum_{i=1}^m S^i$. Then there are $x^i \in S^i$, so that*

$$\left| x - \sum_{i=1}^m x^i \right| \leqslant D(m)^{1/2}.$$

Proof. The proof is by induction on m. The lemma is trivially true for $m=1$. Suppose it is true for $m-1$, we must demonstrate it for m.

$x = u + w$ for some u, w so that $u \in \operatorname{con} \sum_{i=1}^{m-1} S^i$, $w \in \operatorname{con} S^m$. By the inductive hypothesis there are $x^i \in S^i$, $i = 1,..., m-1$ so that $|u - \sum_{i=1}^{m-1} x^i| \leqslant D(m-1)^{1/2}$. Then by the lemma there is $x^m \in S^m$ so that $|w - x^m| \leqslant d(S^m) \leqslant D$ and $(x^m - w) \cdot (\sum_{i=1}^{m-1} x^i - u) \leqslant 0$. We then have

$$\begin{aligned}
&\left| \sum_{i=1}^m x^i - (u+w) \right|^2 \\
&\quad = \left(\sum_{i=1}^m x^i - (u+w) \right) \cdot \left(\sum_{i=1}^m x^i - (u+w) \right) \\
&\quad = \left[\left(\sum_{i=1}^{m-1} x^i - u \right) + (x^m - w) \right] \cdot \left[\left(\sum_{i=1}^{m-1} x^i - u \right) + (x^m - w) \right] \\
&\quad = \left| \sum_{i=1}^{m-1} x^i - u \right|^2 + |x^m - w|^2 + (x^m - w) \cdot \left(\sum_{i=1}^{m-1} x^i - u \right) \\
&\quad \leqslant \left| \sum_{i=1}^{m-1} x^i - u \right|^2 + |x^m - w|^2 \leqslant D^2 m. \qquad \text{Q.E.D.}
\end{aligned}$$

COROLLARY. $r(\sum_{i=1}^m S^i) \leqslant D(m)^{1/2}$.

The Corollary's bound is not so tight as that of the Shapley–Folkman theorem and corollaries, where it is shown that $r(\sum_{i=1}^m S^i) \leqslant (1/2)\, D(\min(m, N))^{1/2}$. It is derived, however, from a peculiarly simple analytic basis and proof. The Proposition and Corollary are similar to Theorem 1 of [4]. The use of statistical independence there corresponds to the possible orthogonality of $(x^m - w)$ to $(\sum_{i=1}^{m-1} x^i - u)$ above. The sequential structure of the analysis here may make this approach particularly suitable for computational use.

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