Lecture 1: Existence of general equilibrium in an economy with an excess demand function

General equilibrium theory focuses on finding market clearing (equilibrium) prices for all goods simultanously. Since there are distinctive interactions across markets the equilibrium concept includes the simultaneous joint determination of equilibrium prices.

N goods in the economy.

A typical array of prices is an N-dimensional vector

$$p = (p_1, p_2, p_3, \ldots, p_{N-1}, p_N) = (3, 1, 5, \ldots, 0.5, 10).$$

Assume only relative prices (price ratios) matter here, not monetary prices.

The price space: The unit simplex in $\mathbf{R}^N$, is

$$P = \left\{ p \mid p \in R^N, p_i \geq 0, i = 1, \ldots, N, \sum_{i=1}^{N} p_i = 1 \right\}. \tag{5.1}$$

The unit simplex is a (generalized) triangle in N-space.

For each household $i \in H$, we define a demand function, $D^i : P \to \mathbf{R}^N$.

For each firm $j \in F$, $S^j : P \to \mathbf{R}^N$.

Positive co-ordinates in $S^j(p)$ are outputs, negative co-ordinates are inputs. $p \cdot S^j(p) \equiv \sum_{n=1}^{N} p_n S_n^j(p) \equiv$ profits of firm j.

The economy has an initial endowment of resources $r \in \mathbf{R}^N$ that is also supplied to the economy.

The market excess demand function is defined as

$$Z(p) = \sum_{i \in H} D^i(p) - \sum_{j \in F} S^j(p) - r, \tag{5.2}$$

$$Z : P \to \mathbf{R}^N \tag{5.3}$$

$Z(p) \equiv (Z_1(p), Z_2(p), Z_3(p), \ldots, Z_N(p))$, where $Z_k(p)$ is the excess demand for good k. When $Z_k(p)$, the excess demand for good k, is negative, we will say that good k is in excess supply.

We will assume the Walras' Law and Continuity of $Z(p)$:

Walras' Law: For all $p \in P$,

$$p \cdot Z(p) = \sum_{n=1}^{N} p_n \cdot Z_n(p) = \sum_{i \in H} p \cdot D^h(p) - \sum_{j \in F} p \cdot S^j(p) - p \cdot r = 0.$$

The economic basis for Walras' Law involves the assumption of scarcity and the structure of household budget constraints. $\sum_{i \in H} p \cdot D^h(p)$ is the value of aggregate household expenditure. The term $\sum_{j \in F} p \cdot S^j(p) + p \cdot r$ is the value of aggregate household income (value of firm profits plus the value of endowment). The Walras Law says that expenditure equals income.

Continuity:

$$Z : P \to \mathbf{R}^N, Z(p) \text{ is a continuous function for all } p \in P.$$

That is, small changes in p result in small changes in $Z(p)$.

We assume in this lecture that $Z(p)$ is well defined and fulfills Walras' Law and Continuity. As mathematical theorists, part of our job is to derive these properties from more elementary properties during the rest of the course (so that we can be sure of their generality).

Definition $p^o \in P$ is said to be an equilibrium price vector if $Z(p^o) \leq 0$ (0 is the zero vector; the inequality applies coordinatewise) with $p^o_k = 0$ for k such that $Z_k(p^o) < 0$. That is, p^o is an equilibrium price vector if supply equals demand in all markets (with possible excess supply of free goods).

Theorem 5.1 (Brouwer Fixed-Point Theorem) Let $f(\cdot)$ be a continuous function, $f : P \to P$. Then there is $x^* \in P$ so that $f(x^*) = x^*$.

Theorem 5.2 Let Walras' Law and Continuity be fulfilled. Then there is $p^* \in P$ so that p^* is an equilibrium.

Proof

Let $T : P \to P$, where $T(p) = (T_1(p), T_2(p), \ldots, T_k(p), \ldots, T_N(p))$. $T_k(p)$ is the adjusted price of good k, adjusted by the auctioneer trying to bring supply and demand into balance. Let $\gamma^k > 0$; γ^k has the dimension, $1/k$. The adjustment process of the kth price can be represented as $T_k(p)$, defined as follows:

$$T_k(p) \equiv \frac{\max[0, p_k + \gamma^k Z_k(p)]}{\sum_{n=1}^{N} \max[0, p_n + \gamma^n Z_n(p)]}. \tag{5.4}$$

The function T is a price adjustment function. It raises the relative price of goods in excess demand and reduces the price of goods in excess supply while keeping the price vector on the simplex. In order for T to be well defined, the denominator must be nonzero, that is,

$$\sum_{n=1}^{N} \max[0, p_n + \gamma^n Z_n(p)] \neq 0. \tag{5.5}$$

(5.5)follows from Walras' Law. For the sum in the denominator to be zero or negative, all goods would have to be in excess supply simultaneously, which is contrary to our notions of scarcity and– it turns out– to Walras' Law as well. Recall that $Z(\cdot)$ is a continuous function. The operations of max[ ], sum, and division by a nonzero continuous function maintain continuity. Hence, $T(p)$ is a continuous function from the simplex into itself.

By the Brouwer Fixed-Point Theorem there is $p^* \in P$ so that $T(p^*) = p^*$.

We must show that p^* is not just the stopping point of the price adjustment process, but that it actually does represent general equilibrium prices for the economy.

Since $T(p^*) = p^*$, for each good k, $T_k(p^*) = p^*_k$. That is, for all $k = 1, \ldots, N$,

$$p^*_k = \frac{\max[0, p^*_k + \gamma^k Z_k(p^*)]}{\sum_{n=1}^{N} \max[0, p^*_n + \gamma^n Z_n(p^*)]}. \tag{5.6}$$

Either

$$p_k^* = 0 \qquad \text{(Case 1)} \tag{5.7}$$

or

$$p_k^* = \frac{p_k^* + \gamma^k Z_k(p^*)}{\sum_{n=1}^{N} \max[0, p_n^* + \gamma^n Z_n(p^*)]} > 0 \qquad \text{(Case 2).} \tag{5.8}$$

CASE 1 $p_k^* = 0 = \max[0, p_k^* + \gamma^k Z_k(p^*)]$. Hence, $0 \geq p_k^* + \gamma^k Z_k(p^*) = \gamma^k Z_k(p^*)$ and $Z_k(p^*) \leq 0$. This is the case of free goods with market clearing or with excess supply in equilibrium.

CASE 2 To avoid repeated messy notation, let

$$\lambda = \frac{1}{\sum_{n=1}^{N} \max[0, p_n^* + \gamma^n Z_n(p^*)]} \tag{5.9}$$

so that $T_k(p^*) = \lambda(p_k^* + \gamma^k Z_k(p^*))$. Since p^* is the fixed point of T we have $p_k^* = \lambda(p_k^* + \gamma^k Z_k(p^*)) > 0$. This expression is true for all k with $p_k^* > 0$, and λ is the same for all k. Let's perform some algebra on this expression. We first combine terms in p_k^*:

$$(1 - \lambda)p_k^* = \lambda \gamma^k Z_k(p^*), \tag{5.10}$$

then multiply through by $Z_k(p^*)$ to get

$$(1 - \lambda)p_k^* Z_k(p^*) = \lambda \gamma^k (Z_k(p^*))^2, \tag{5.11}$$

and now sum over all k in Case 2, obtaining

$$(1 - \lambda) \sum_{k \in \text{Case2}} p_k^* Z_k(p^*) = \lambda \sum_{k \in \text{Case2}} \gamma^k (Z_k(p^*))^2. \tag{5.12}$$

Walras' Law says

$$0 = \sum_{k=1}^{N} p_k^* Z_k(p^*) = \sum_{k \in \text{Case1}} p_k^* Z_k(p^*) + \sum_{k \in \text{Case2}} p_k^* Z_k(p^*). \tag{5.13}$$

But for $k \in$ Case 1, $p_k^* Z_k(p^*) = 0$, and so

$$0 = \sum_{k \in \text{Case1}} p_k^* Z_k(p^*). \tag{5.14}$$

Therefore,

$$\sum_{k \in \text{Case2}} p_k^* Z_k(p^*) = 0. \tag{5.15}$$

Hence, from (5.11) we have

$$0 = (1 - \lambda) \cdot \sum_{k \in \text{Case2}} p_k^* Z_k(p^*) = \lambda \cdot \sum_{k \in \text{Case2}} \gamma^k (Z_k(p^*))^2. \tag{5.16}$$

Using Walras' Law, we established that the left-hand side equals 0, but the right-hand side can be zero only if $Z_k(p^*) = 0$ for all k such that $p_k^* > 0$ (k in Case 2). Thus, p^* is an equilibrium. This concludes the proof.

QED