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PROPORTIONAL DISTRIBUTION SCHEMES

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An amount of income can be obtained jointly by m agents, the ith agent's *share* of income being θ_i. The income and the utilities of each agent are functions of the state of nature. Each agent has a probability measure over the states of nature. An *efficient proportional distribution* is one which is (1) Pareto optimal and for which (2) the expected proportion of income agent i receives divided by θ_i is independent of i. It is shown that if the attitudes are strictly concave then there exists exactly one proportional distribution scheme. Furthermore, in special cases, each agent expects to receive an income that exceeds his share.

1. Introduction

The model presented here deals with a simple economy in which a number of agents desire to distribute an amount of income that depends on an uncertain event. Each agent i is assumed to have a utility function $u_i(s, c)$ which depends on income c as well as the state of nature s. Each agent also has a probability distribution μ_i, which reflects his expectations.

The approach of this paper is to classify the distribution schemes that are *efficient* (or optimal), those schemes that have the property that no other scheme can provide a higher expected utility to all of the agents, in terms of the expected share of income an agent receives. The main result is that given any positive numbers θ_i (shares) summing to one there is exactly one efficient distribution with the property that θ_i/θ_j is equal to the ratio of the expected proportions of income received by i to that of j. Thus, if $x_i(s)$ is the income agent i receives in state s and $I(s)$ is the total income in state s, we have $\int x_i(s)\mathrm{d}\mu_i/\theta_i \int I(s)\,\mathrm{d}\mu_i$ independent of i. Such a scheme will be called an *efficient* (*or optimal*) *proportional distribution* corresponding to the shares $\theta_1,\ldots,\theta_m$. When all expectations are the same, proportionality reduces to $\int x_i(s)\mathrm{d}\mu_i = \theta_i \int I(s)\,\mathrm{d}\mu_i$ for all i.

This result provides a complete classification of the efficient states of the economy. It is well known that the efficient points of an economy whose agents have strictly concave utility functions can be identified with a simplex of appropriate dimension. Typically the characterization results from the

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observation that efficient allocations maximize a weighted average of utility functions; efficient allocations can be associated with these weights. Here the correspondence is between efficient allocations and expected shares.

If there is some way to determine the shares, then the result singles out a unique efficient distribution. For example, if a group of workers with known relative abilities and identical jobs are required to distribute the income derived from their combined labor, the shares could correspond to their relative abilities. In this instance, the state of nature could reflect the income of the firm as well as other factors relevant to the workers. There is no reason to expect that the expectations of the workers would be identical in such a situation. If the total income I is obtained by adding the income I_i of the agents, we may take $\theta_i = I_i/I$. There are two problems with this approach. First, if income depends on the state of nature, θ_i is not well defined (the expected ratio of I_i to I will differ from agent to agent). Furthermore, if we derive the shares from individual income, the distribution scheme should be *individually rational*: if $x_i(s)$ is the income distributed to agent i, it must satisfy $\int u_i(s, x_i(s))\,\mathrm{d}\mu_i \geqq \int u_i(s, I_i)\,\mathrm{d}\mu_i$. The efficient proportional distribution corresponding to the shares $I_1/I, \ldots, I_m/I$ need not have this property. Individual rationality is not a consideration if an agent is unable to guarantee his share of the income without the aid of other agents. Thus, in the example concerning distribution of income derived from a production process, the lack of individual rationality will not be a problem if the technology exhibits increasing returns to scale or if, for some reason, workers are unable to generate income on their own.

If there is some recognized procedure for determining the shares, the efficient proportional distribution has some appeal as an equitable outcome of the distribution process. However, no additional properties associated with fairness can be guaranteed. In particular, even if all shares are equal, there is no reason to believe that an agent will not prefer (in the expected utility sense) the income another receives. Nevertheless, in certain cases another property can be asserted: that agent i's expected percentage of income will exceed his share. If the income is thought of as a cake to be divided among the agents, then this result describes the situations in which it is possible to efficiently divide up the cake so that each agent believes he is receiving more than his share. Dubins and Spanier (1961) prove a similar result when all of the agents have linear utility functions. In section 7 it is shown that their theorem follows as a special case of the one presented here.

The conclusions of this paper should be compared to several earlier results. Gale and Sobel (1980) presented a model similar to this one. However, agents were assumed to have identical expectations. It is shown that corresponding to any given shares $\theta_1, \ldots, \theta_m$ there is a unique efficient distribution with agent i receiving expected share θ_i for each i. This follows as a special case of the results presented here. On the other hand, Gale and

Sobel proved that the desired allocation can be characterized as the solution to a certain constrained maximization problem. An analogous result has not been found in the more general case presented here, so independent proofs have been provided. The fact that the distribution problem with different expectations cannot be solved by the maximization of a single function can be put in perspective if the results of Wilson (1968) are considered. Wilson was interested in a model more general than ours, in which a group chooses a distribution (sharing rule) and an action that will effect a random outcome. In our model, the group has no explicit influence on the outcome. Wilson shows that unless expectations are identical, or the sharing rule is linear, there will be no way to determine an efficient allocation my maximizing a social utility function satisfying the Savage axioms.

The second section of this paper provides a formal description of the model. Section 3 proves existence and section 4 uniqueness of efficient proportional distributions. In section 5, two special cases are analyzed. In these, each agent expects to receive more than his share in the efficient proportional distribution. Section 6 looks at the special case in which the agents have different expectations, but identical utility functions. In this case, an efficient scheme of 'proportional bets' can be constructed to give positive expected utility to every agent. Finally, section 7 deduces several results of Dubins and Spanier (1961) as special cases of our analysis.

2. The model

In this section we describe the problem. The characteristics of the model are:

(i) a non-negative, bounded, measurable function I (the total income) defined on a probability space S (the states of nature).

(ii) real valued utility functions u_i defined on $S \times R_+$ for $i=1,\dots,m$. The functions are denoted by $u_i(s,c)$ and are assumed to be increasing and concave in c. Letting u_i' denote differentiation with respect to c, we also assume u_i and u_i' are bounded and measurable in s.

(iii) probability measures μ_i representing the expectations of the agents. We assume the measures are mutually absolutely continuous [that is, if $\mu_j(A)=0$ for some j then $\mu_i(A)=0$ for all i] hence the measures can be replaced by bounded measurable functions p_i which can be taken to be positive almost everywhere. The p_i are densities (Radon–Nikodym derivatives) with respect to a common measure, say $\gamma = m^{-1}\sum_{i=1}^{m}\mu_i$, and satisfy $\int_S f(s)\,\mathrm{d}\mu_i = \int_S f(s)p_i(s)\,\mathrm{d}\gamma$ for all integrable functions f. In what follows all integrals will be with respect to the measure γ; reference to γ is suppressed in our notation. Also we write $\int f$ for $\int_S f$. Finally, the p_i are normalized so that $\int Ip_i = 1$ for each i.

(iv) positive numbers $\theta_1,\dots,\theta_m$ summing to one (the shares). If the total income is independent of the state of nature, and is obtained as the sum of individual incomes I_i, we can take $\theta_i = I_i/I$. Notice that the shares cannot be defined unambiguously if I_i depends on s. In this case there need not be agreement about what an agent's expected share of income is.

A *distribution scheme* is a vector valued non-negative measurable function $d=(d_1,\dots,d_m)$ on S; $d_i(s)$ represents the consumption distributed to agent i in state s. We denote by D the set of all distribution schemes. The scheme d in D is called *feasible* if

$$\sum d_i(s) = I(s) \quad \text{for all } s.$$

If in addition we have

$$\left(\int d_i(s) p_i(s)\right)/\theta_i \quad \text{independent of } i,$$

we say that d is *proportional.* That is, a feasible distribution scheme is proportional provided that agent i's expected proportion of income divided by his share is independent of i. Because of the particular normalization chosen for p_i, $\int d_i(s)p_i(s)$ is equal to the expected share of income received by agent i. That is, $\int d_i(s)p_i(s) = \int d_i(s)p_i(s)/\int I(s)p_i(s)$.

For each d in D we define the m-vector $u(d)$ by

$$u(d) = \left(\int u_1(s, d_1(s))p_1(s), \dots, \int u_m(s, d_m(s))p_m(s)\right).$$

Thus $u(d)$ is the set of expected utilities of the agents under the scheme d. A feasible scheme $\bar{d}$ is called *optimal* or *efficient* if there is no other feasible scheme d such that $u(d) > u(\bar{d})$.

3. Existence of optimal proportional distributions

In this section we will show that efficient proportional distributions exist. The proof is a consequence of the next lemma:

Lemma. Let α be a convex valued upper hemicontinuous correspondence from the $(m-1)$-dimensional simplex to itself. If $\lambda_i = 0$ implies $v_i = 0$ for all $v \in \alpha(\lambda)$, then α is surjective.

When α is single-valued this is a consequence of Brouwer's Theorem. In the general case one can construct a proof by using an approximation theorem. For example, the result of Celina given in Hildenbrand and Kirman (1976, app. IV) is sufficient to prove the lemma.

Now define a correspondence ψ from the $(m-1)$-dimensional simplex Δ to the set of feasible distribution schemes $\hat{D}$ by

$$\psi(\lambda)=\{d: d \text{ solves } \max \sum \lambda_i \int u_i(s, x_i(s))p_i(s),$$
$$\text{subject to } \sum v_i(s)=I(s)\}.$$

Notice that if $d \in \psi(\lambda)$ for some λ then x is efficient. Next define $f: \hat{D} \to \Delta$ by

$$f(d)=(\int d_1(s)p_1(s), \dots, \int d_m(s)p_m(s))/\sum \int d_i(s)p_1(s).$$

It is routine to check that under our assumptions on the utility functions the correspondence $\alpha = f \circ \psi$ is well defined and satisfies the assumptions of the lemma [a similar verification is given in Gale and Sobel (1980)]. Therefore there exists $\lambda \in \Delta$ with $\theta \in \alpha(\lambda)$. Thus if $d \in \psi(\lambda)$ then d is an efficient proportional distribution. We have established:

Theorem 1. There exists an efficient proportional distribution corresponding to the given shares $\theta_1, \dots, \theta_m$.

4. Uniqueness of efficient proportional distributions

The proof of uniqueness depends on an analysis of the necessary conditions for optimality.

Let $\bar{x}$ be an optimal proportional distribution. Then there is a $\lambda \in \Delta$ such that $\bar{x}$ solves

$$\max \sum \lambda_i \int u_i(s, x_i(s))p_i(s), \qquad \text{subject to } \quad \sum x_i(s)=I(s). \tag{1}$$

This is a problem considered in Gale and Sobel (1980) and therefore we can assert that if $\bar{x}$ is a solution to (1) then there exists a function β satisfying, for almost every s and every i,

$$\lambda_i u_i'(s, \bar{x}_i(s))p_i(s) \leqq \beta(s), \qquad \text{with equality if } \quad \bar{x}_i(s)>0. \tag{2}$$

We can now prove the uniqueness[1] theorem.

[1]We say $x=y$ if $\gamma(\{s: x_i(s) \neq y_i(s)\})=0$ for each i.

Theorem 2.[2] *If $u_i'(s,c)$ is a strictly decreasing function of c for all i then there is at most one efficient proportional distribution associated with the shares $\theta_1,\ldots,\theta_m$.*

Proof. Suppose x and y are two efficient proportional distributions associated with the shares $\theta_1,\ldots,\theta_m$ and that $\int y_i(s)p_i(s) \geqq \int x_i(s)p_i(s)$ for all i. Then it follows from (1) and (2) that there exist λ and μ in Δ such that

$$\frac{u_j'(s,x_j(s))p_j(s)}{u_j'(s,x_i(s))p_i(s)} \geqq \frac{\lambda_i}{\lambda_j} \quad \text{if} \quad x_j(s)>0,$$

$$\frac{\mu_i}{\mu_j} \geqq \frac{u_j'(s,y_j(s))p_j(s)}{u_i'(s,y_i(s))p_i(s)} \quad \text{if} \quad y_i(s)>0.$$

Moreover, by relabeling the agents if necessary, λ_i/u_i can be taken to be non-increasing in i. Thus, for any i and $j>i, x_j(s)>0$ and $y_i(s)>0$ imply

$$\frac{u_j'(s,x_j(s))}{u_i'(s,x_i(s))} \geqq \frac{u_j'(s,y_j(s))}{u_i'(s,y_i(s))}.$$

Hence, since $u_i'(s,c)$ is decreasing in c,

$$y_i(s)>x_i(s) \quad \text{implies} \quad y_j(s) \geqq x_j(s) \quad \text{for all} \quad j>i. \tag{3}$$

In order to obtain a contradiction, suppose that $x \neq y$ and let k be the smallest integer with $y_k \neq x_k$. It follows from (3) and the fact that $\sum x_i = \sum y_i$ that $y_k \leqq x_k$ with strict unequality on a set of positive measure. As this contradicts $\int y_k(s)p_k(s) \geqq \int x_k(s)p_k(s)$, the theorem is established. ▮

5. Properties of proportional distributions

Suppose $\bar{x}$ is an efficient proportional distribution. In order to have appeal as an equitable outcome of the distribution process, $\bar{x}$ should have some additional properties. For example, if expectations are different it seems reasonable to expect that $\int p_i\bar{x}_i > \theta_i$ for each i. That is, each agent expects to receive more than his objective share. Unfortunately, this may not be the case in general. For example, in a two-agent, two-state world, with total income of one unit in each state, let agent one have probabilities (1/2, 1/2) and utility function $u_1(j,c)=jc$ for $j=1,2$, and let agent two have probabilities (1/3, 2/3) and $u_2(j,c)=c/j$ for $j=1,2$. Then the unique proportional distribution with shares (1/2, 1/2) is (0, 6/7) for agent one and

[2] I am grateful to Hilton Machado for discovering an error in an earlier version of this proof.

(1, 1/7) for agent two. Thus each agent expects to receive 3/7 of the total income.

What makes the above example possible is a difference in tastes that more than compensates for different expectations. This possibility can be ruled out in two special cases:

Theorem 3. Let $\bar{x}$ *be an efficient proportional distribution and let* $K = \int p_i \bar{x}_i / \theta_i$. *If*

(A) $u_i = u$ *and* $\theta_i = 1/m$ *for all i, or*

(B) $u_i(s, c) = u_i(c)$ *for all i* and $I(s)$ *is independent of s,*

then $K \geqq 1$.

Observe that $K \geqq 1$ is equivalent to $\int p_i \bar{x}_i \geqq \theta_i$ for each i. Condition (A) implies that each agent is identical with respect to all characteristics except expectations. This corresponds to a situation in which similar workers have different attitudes about the future. For condition (B) to hold all preferences must be independent of the state of nature. This situation may be interpreted as a gambling model where each agent has I_i dollars to bet and the $x_i(s)$ is the amount received by i in the event s. The result then guarantees that each agent has non-negative expected winnings (if $\theta_i = I_i / \sum_j I_j$).

Proof. Suppose $\bar{x}$ is efficient. Then by (2) there exists a function β satisfying, for almost every s,

$$\lambda_i p_i(s) \leqq \beta(s)/u_i'(s, \bar{x}_i(s)), \qquad \text{with equality if} \quad \bar{x}_i(s) > 0. \tag{4}$$

It follows that, for almost every s,

$$\lambda_i p_i(s) \bar{x}_i(s) = \beta(s) \bar{x}_i(s)/u_i'(s, \bar{x}_i(s)). \tag{5}$$

Suppose condition (A) holds. Integrating both sides of (5) and summing over i yields

$$\left(\sum \frac{1}{m} \lambda_i\right) K = \int \beta(s) \left[\sum \frac{\bar{x}_i(s)}{u'(s, \bar{x}_i(s))}\right]. \tag{6}$$

But, for fixed s,

$$\sum \frac{\bar{x}_i(s)}{u'(s, \bar{x}_i(s))} \geqq \frac{I(s)}{m} \sum \frac{1}{u'(s, \bar{x}_i(s))}, \tag{7}$$

since $\sum (\bar{x}_i(s) - I(s)/m) = 0$ and $1/u'(s, c)$ is an increasing function of c for fixed s.

Combining (6) and (7), and an application of (4) yield

$$\left(\sum \frac{1}{m}\lambda_i\right)K \geqq \frac{1}{m}\int \sum \frac{\beta(s)I(s)}{u'(s,\bar{x}_i(s))} \geqq \frac{1}{m}\int \sum \lambda_i p_i(s)I(s) = \sum \frac{1}{m}\lambda_i,$$

and so $K \geqq 1$.

To prove (B), observe that if $v_i = \int p_i(s)u_i'(\bar{x}_i(s))$, then

$$\int [v_i\theta_i KI - p_i(s)\bar{x}_i(s)u_i'(\bar{x}_i(s))] \geqq 0. \tag{8}$$

This follows since

$$\int [v_i\theta_i KI - p_i(s)\bar{x}_i(s)u_i'(\bar{x}_i(s))]$$

$$= \int [Ip_i(s)v_i - p_i(s)u_i'(\bar{x}_i(s))]\bar{x}_i(s)$$

$$\geqq \int_{A_i} [Ip_i(s)v_i - p_i(s)u_i'(\bar{x}_i(s))]\bar{x}_i(s) + \int_{S-A_i} [Ip_i(s)v_i - p_i(s)u_i'(\bar{x}_i(s))]\bar{x}_i(s)$$

$$\geqq c_i \int [I_{p_i}(s)v_i - p_i(s)u_i'(\bar{x}_i(s))] = 0,$$

where $c_i > 0$ satisfies

$$Iv_i = u_i'(c_i) \quad \text{and} \quad A_i = \{s : Iv_i > u_i'(\bar{x}_i(s))\}.$$

Integrating (5) and using (8) we get

$$\theta_q\lambda_q v_q KI \geqq \int \beta(s)\bar{x}_q(s) \quad \text{for each } q.$$

Summing and using (4) gives, for each i,

$$\left(\sum_q \theta_q\lambda_q v_q\right)KI \geqq I\int \beta(s) \geqq \lambda_i I \int p_i(s)u_i'(\bar{x}_i(s)) = \lambda_i v_i I.$$

Thus,

$$K = \lambda_i v_i \Big/ \sum_q \theta_q\lambda_q v_q \quad \text{for all } i,$$

and so $K \geqq 1$, the desired result. ∎

Some interpretation of the proof of condition (B) is possible. Notice that, roughly speaking, $p_i(s)u_i'(\bar{x}_i(s))$ is a price system supporting the efficient allocation $\bar{x}$. Then (8) says that the value of the income $\bar{x}_i$ is no more than the product of average prices (v_i) and average income $(\theta_i KI)$ for every agent. Adding these inequalities yields the conclusion $K \geqq 1$.

Also notice that the inequalities in (7) and (8) will hold strictly provided agents do not have identical expectations. Hence, $K > 1$ in this case.

6. Proportional bets

In view of the results of the previous sections we can reformulate our model to describe a special case. Suppose that each agent i has a positive amount of money β_i and a utility function v_i satisfying our boundedness, smoothness and concavity assumptions, and depending on income alone. Then, provided the agents have different expectations, there will be an incentive to gamble.

Call a vector $y=(y_1,\ldots,y_m)$ of measurable functions from S to R a *bet* if $y_i(s) \geqq -\beta_i$ and $\sum_{i=1}^m y_i(s)=0$. By defining functions u_i by $u_i(c)=v_i(c-\beta_i)$, letting $I=\sum_{i=1}^m \beta_i$, and $\theta_i=\beta_i/I$, we can apply the theorems of sections 3 and 4 to assert the existence and uniqueness of an efficient proportional distribution $\bar{x}=(\bar{x}_1,\ldots,\bar{x}_m)$. Furthermore, by Theorem 3(B) we have $\int \bar{x}_i(s)p_i(s) \geqq \theta_i$ and, letting $\bar{y}_i(s)=\bar{x}_i(s)-\beta_i$, an efficient bet with the expected shares $\int \bar{y}_i/\theta_i$ independent of i, and expected winnings non-negative for every agent.

7. Linear utility functions

Dubins and Spanier (1961) discuss a model of division for the case in which all utility functions are linear, $u_i(s,c)=c$, and $I(s)=1$ for all s. In this section we deduce one of their results as a consequence of our analysis.

Theorem 4. If the measures μ_i are non-atomic and $u_i(s,c)=c$ for all i, then there is a partition $\{A_1,\ldots,A_m\}$ of S such that the distribution $\bar{x}$, defined by

$$\bar{x}_i(s)=I_i(s) \quad \textit{for} \quad s \in A_i,$$

$$=0 \qquad \textit{for} \quad s \in A_i,$$

is efficient, and $\int p_i(s)\bar{x}_i(s)/\theta_i$ is independent of i.

Proof. Let x be an efficient proportional distribution. For every non-empty subset T of $\{1,\ldots,m\}$ let $S_T=\{s: x_i(s)>0$ if and only if $i \in T\}$. Thus S_T is the set of states for which each agent in subset T gets positive income and everyone not in T receives 0 income. Then for every T, S_T can be partitioned into $A_i(T)$ with

$$\int_{A_i(T)} I_i(s)p_i(s) = \int_{S_T} x_i(s)p_i(s). \tag{9}$$

To see this notice that (2) implies that if $i,j \in T$ then there exist positive numbers λ_i, λ_j with

$$\lambda_i p_i(s) = \lambda_j p_j(s) \quad \text{for almost every} \quad s \in S_T. \tag{10}$$

Order the elements of $T, i_1, \ldots, i_r$, and suppose that $A_{i_1}(T), \ldots, A_{i_{k-1}}(T)$ are disjoint subsets of S_T satisfying (9). Then, using (10),

$$\begin{aligned}\lambda_{i_k} \int_{S_T - \sum_{q<k} A_{i_q}(T)} p_{i_k} I &= \lambda_{i_k} \int_{S_T} p_{i_k} I - \sum_{q<k} \lambda_{i_q} \int_{A_{i_q}(T)} p_{i_q} I \\ &= \lambda_{i_k} \int_{S_T} p_{i_k} I - \sum_{q<k} \lambda_{i_q} \int_{S_T} p_{i_q} x_i \\ &= \lambda_{i_k} \int_{S_T} p_{i_k} \left(I - \sum_{q<k} x_{i_q} \right) \\ &\leqq \lambda_{i_k} \int_{S_T} p_{i_k} x_{i_k}.\end{aligned}$$

Hence, since μ_{i_k} is non-atomic, a set $A_{i_k}(T)$ can be chosen to satisfy (9). By induction, this process can be continued until $A_i(T)$ is defined for all $i \in T$. Since x is efficient we may assume $\bigcup_{i \in T} A_i(T) = S_T$. Finally, the claim is established by setting $A_i(T) = \phi$ for $i \notin T$. The theorem follows by defining A_i by $A_i = \bigcup_T A_i(T)$. ∎

The above theorem states that we can choose the distribution to be a partition provided all utility functions are linear and measures non-atomic. If, in addition, $I(s)$ is constant we can apply condition (B) of Theorem 3 to assert the existence of a partition $\{A_1, \ldots, A_m\}$ with $u_i(A_i)/\theta_i$ independent of i, and $u_i(A_i) \geqq \theta_i$ for every i. This demonstrates the existence of what are called 'equitable' partitions in Dubins and Spanier (1961).

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