Econ 172A, Fall 2014: Final Examination Solutions (I)

Comments on Grades.

1. I determined your course grade by adding your final exam score (out of 160) and your midterm score (out of 112) to twice the sum of your three best quizzes (out of 48) to get a score out of 320. From this number I assigned letter grades. The cutoff for various grades was: A: 250; A-: 242; B+: 238; B: 208; B-: 195; C+: 192; C: 172; C-: 155; D: 140. (I did make adjustments near the cutoffs for special cases.)

2. The graded finals will be available in the Economics Department 245 SH (if you did not sign the Buckley waiver) or outside the Economics Department Building near Econ 108 (if you signed the waiver).

3. Information about the grade distribution should be available on TED. (Briefly, for the final: 160 were possible, 154 high, 106 median, 25 low; for the class: 320 possible; range: 32–304, median 214.)

Comments on Final The first four questions went quite well and Question 6 was reasonably good. Question 7 was a disaster. I was disappointed in the answers to Question 5. We awarded points generously to people who showed evidence of having the right idea.

1. Consider a stable marriage problem in which there are five men, (1, 2, 3, 4, 5), and five woman (A, B, C, D, E).

   Suppose preferences are given by the following tables:

   Mens' Preferences

MEN					
1	C	D	E	B	A
2	E	A	D	B	C
3	B	A	C	E	D
4	B	E	D	A	C
5	A	C	D	E	B

   Womens' Preferences

WOMAN					
A	5	2	3	1	4
B	1	4	3	5	2
C	2	5	3	1	4
D	4	1	5	2	3
E	5	3	2	1	4

   So, for example, Man 1 prefers Woman C to Woman D, Woman D to Woman E, Woman E to Woman B, and Woman B to Woman A.

   Exhibit two stable marriages for this problem. Justify your answer. (One way to solve the problem is to apply the algorithm from class twice, once with the men making offers and once with the women making offers.)

   FIrst assume that men propose. First round: 1-C,2-E,3-B,4-B,5-A. Second Round: A holds 5, B holds 4, rejects 3, C holds 1, D alone, E holds 2. Third Round: 3-A. Fourth Round: A holds 5, rejects 3; B holds 4; C holds 1; D alone; E holds 2. Fifth round: 3-C. Sixth round: A holds 5; B holds 4; C holds 3, rejects 1; D alone; E holds 2. Seventh round: 1-D. Eight round: A-5; B-4; C-3; D-1; E-2. Stable matching.

   Next assume that women propose: First round: A-5,B-1,C-2,D-4,E-5. Second round: 1-B,2-C,3-alone,4-D,5-A (5 rejects E). Third round: E-3. Fourth round: 1-B, 2-C, 3-E, 4-D, 5-A.

2. Refer to the diagram at the end of the examination on the page headed: Problem 2, Form 1: Minimum Spanning Tree. The numbers on the edges are costs. Note the cost on the edge connecting D to E is x.

   (a) Assume $x = 4$. Find a minimal spanning tree for the network. (You may draw the minimum spanning tree below or state which edges are in the tree.) What is the cost of the minimum spanning tree? Is the minimum spanning tree unique? If yes, explain why. If no, identify another minimum spanning tree.

   MST: A-B; B-E, C-E, E-G, F-G and either A-D or D-E (not unique). Cost $2 + 1 + 1 + 1 + 2 + 4 = 11$. One obtains this answer by following an algorithm (there are two). One is to keep adding the cheapest edges provided that you don't form a cycle. You add the three edges that cost 1, followed by the edges that cost 2. Adding either 3 forms a cycle. The last edge is either A to D or D to E.

   (b) For what values of x does there exist a minimal spanning tree that includes the edge connecting D to E?

   $x \leq 4$. The previous answer shows that it is part of MST when $x = 4$. If x decreases, it would still be part of MST. If x increases, it would be better to include A to D instead of D to E.

3. Refer to the diagram at the end of the examination headed: Problem 3, Form 1: Shortest Route. The numbers on the edges are costs (distances). The edges are directed (have arrows). You can only travel in the direction of the arrow.

   Use the algorithm introduced in class to find the shortest route between node A and node F. Identify the shortest route and its cost. Use the table below to show your work. Note: To receive credit, you must use the algorithm discussed in class.

Iteration	A	B	C	D	E	F	G
1							
2							
3							
4							
5							
6							

Iteration	A	B	C	D	E	F	G
1	0*	2**	∞	8	5	∞	∞
2	0*	2*	3**	8	5	∞	∞
3	0*	2*	3*	8	4**	∞	9
4	0*	2*	3*	8	4*	9	6**
5	0*	2*	3*	8	4*	7**	6*
6	0*	2*	3*	8**	4*	7*	6*

   Shortest route: $A \to B \to C \to E \to G \to F$. Cost 7.

4. Refer to the diagram at the end of the examination headed: Problem 4, Form 1: Maximum Flow. The numbers on the edges are capacities. The edges are directed (have arrows). Flows can travel only in the direction of the arrows.

   (a) Find a maximum flow (from the source node to the sink node) for this network. You may describe the flow here or on one of the network diagrams on the next pages. Please indicate your answer clearly. You must explain how you found the answer. If you properly used the algorithm introduced in class (and you show the steps), then you need no further justification. If you used another method, then you must explain the method and explain how it works.

   (b) Find the value of the maximum flow.

   50, see attached.

   (c) Find a minimum capacity cut for this network.

   $\{\text{source}, A, C, D, G, H\}, \{B, E, F, \text{sink}\}$

   (d) What is the capacity of the minimum capacity cut?

   50

   (e) Find the capacity of the cut $\{\text{source}, A, B, C, D\}$ and $\{E, F, G, H, \text{sink}\}$.

   $70 + 10 + 15 + 5 + 30 + 20 = 150$

   (f) Suppose that you could create an "Improved Network" by increasing the flow capacity of exactly one edge by ten units. Your objective is to increase the maximum flow by as much as possible. On which edge would you increase capacity and what would be the value of the maximum flow in the improved network?

   Increasing capacity on C to E or D to E by 10 will increase the flow by 10. I think that these are the only ways to improve things for the solution I found.

5. Pat (a farmer) grows oranges. Pat can sell oranges in bags or as orange juice. Oranges are graded on a quality scale from 1 (poor) to 10 (perfect). Assume that Pat has S_j oranges of Quality j (for $j = 1, 2, \dots, 10$). The average quality of oranges sold in bags must be at least seven. The average quality of the oranges used in orange juice must be at least eight. (Otherwise the contents of bags of oranges and orange juice are unconstrained.) Each pound of oranges that is used for juice yields a profit of \$.80. Each pound of oranges sold in bags yields a profit of .45. Pat wishes to find a production plan (the number of pounds of oranges of each quality to put into bags or to sell as orange juice) that meets average quality restrictions and maximizes profit using no more than his available supply of orange.

   What follows is a formulation. I will say that a "bag of orange" is "Product 1" and "orange juice" is "Product 2." I let x_{ij} = the number of pounds of quality j orange sold as product i. Here $i = 1$ or 2 and $j = 1, 2, \dots, 10$. The goal is to find x_{ij} to solve:

   $\max .45 \sum_{j=1}^{10} x_{1j} + .8 \sum_{j=1}^{10} x_{2j}$ subject to:

   (1) $x_{1j} + x_{2j} \leq S_j$ for $j = 1, 2, \dots, 10$,

   (2) $\sum_{j=1}^{10} j x_{1j} \geq 7(\sum_{j=1}^{10} x_{1j})$,

   (3) $\sum_{j=1}^{10} j x_{2j} \geq 8(\sum_{j=1}^{10} x_{2j})$,

   (4) $x_{ij} \geq 0$ for all $j = 1, 2, \dots, 10$ and $i = 1, 2$.

   (a) How many variables do I use in this formulation?

   20 (2 products times ten quality levels)

   (b) Not counting non-negativity constraints, how many constraints are there? (That is, how many different conditions are described in Lines (1), (2), and (3).) Hint: The answer is not three.

   12 (Line 1 contains ten constraints; the other two lines are one constraint each.)

   (c) Explain Constraint (2) in words. What does the left-hand side represent? What does the right-hand size represent? What is the relationship between the algebraic expression and the statement of Pat's problem?

   The left hand side is the "total quality" of Product 1: the sum of the amount of various oranges weighted by quality. The right hand side is the total (unweighted) quantity of Product 1 times the minimum allowable quantity. If you divide both sides by the total quantity, the constraint becomes "average quality of Product 1" (on left) greater than or equal to 7. This is the quality constraint for Product 1 stated in the problem.

   (d) Rewrite Constraint (2) so that it is in the form:

$$\sum_{j=1}^{10} a_{1j} x_{1j} \leq b_1.$$

$$\sum_{j=1}^{10} (7-j) x_{1j} \leq 0.$$

I used Excel to solve a special case of this problem. In the special case, Pat has 100,000 pounds of Quality 6 and 180,000 pounds of Quality 9 available (but no supply of any other quality). Using the Excel information attached, answer the following questions.

(e) What is the solution to Pat's problem? That is, how much of each product does Pat produce and how many pounds of orange of each quality does Pat use in each product?

Pat uses 13,333.33 pounds of Quality 6 Oranges and 6,666.67 pounds of Quality 9 Oranges to make a total of 20,000 pounds of the first product. Pat uses 86,666.67 pounds of Quality 6 Oranges and 173,333.33 pounds of Quality 9 Oranges to make a total of 260,000 pounds of the second product.

(f) What is Pat's profit?

$ 217000

(g) How would Pat's profit change if Pat had 10,000 fewer pounds of Quality 6 oranges? (Answer as completely as possible.)

Shadow price .1, within allowable range, so profits decrease by $1,000.

(h) How would Pat's profit change if Pat had 210,000 pounds of Quality 9 oranges available? (Answer as completely as possible.)

Increase of 30,000 is outside of the allowable range. We know that the first 20,000 (upper bound) lead to an increase of $ 1.15 each. Hence profits go up by at least $ 23,000. We can say more. Pat can always sell a Quality 9 orange as juice and make 85 cents per pound. So the remaining oranges must be worth at least $8, 500 and profits go up by at least $ 31,500.

(i) Suppose that Pat can market a lower-quality orange juice to UCSD Dining Services. This juice must use oranges that are average quality of 7.5 or higher. At what profit level per pound would Pat be willing to supply this new product (using his existing resources)?

Quality 7.5 is really a one to one mixture of Quality 6 and Quality 9 oranges. Hence the (shadow) cost of the new juice is the average of the shadow prices of the two capacity constraints: $1.25/2 = .625$. Hence Pat would be willing to sell the new product for anything over 62.5 cents per pound.

(j) How would your answer to the previous part change if the new juice had to have average quality of 9 or higher?

The new product would need to be made exclusively of Quantity 9 oranges. Hence it would need to sell for the shadow price of the Quantity 9 constraint, \$ 1.15 per pound.

(k) Do you know how much Pat would be willing to pay for an pound of oranges with Quality 7? Explain your answer carefully.

A Quality 7 orange is equivalent to a mixture of 2/3 Quality 6 and 1/3 Quality 9, so the value is $1/3(.1 + 1.15) = .45$. Pat should be willing to pay 45 cents for a pound of Quality 7 oranges.

6. Suppose that you have a connected network with 12 nodes. Each edge in the network has a weight between 1 and 10.

   (a) What is the smallest value of a minimal spanning tree for this network.

   11

   (b) What is the greatest value of a minimal spanning tree for this network.

   110

   (c) If the weight on one edge decreased by one, is it possible for the value of the minimum spanning tree to stay the same? Decrease by exactly one? Decrease by more than one? Explain.

   It could stay the same (the edge could have had an extremely high weight, so it is not in the MST before or after the change). It could decrease by exactly one if the edge is in the MST. It cannot decrease by more than one (because if it did, the tree you find for the new problem would be better than the solution to the original problem when evaluated using the original weights).

7. Suppose that x^* is the solution to

$$\max c \cdot x \text{ subject to } Ax \leq b, x \geq 0$$

and each component of Ax^* is strictly less than the corresponding component of b. Can the value of the problem $(c \cdot x^*)$ be positive? Can the value be negative? Explain.

The value must be 0 by Complementary Slackness. (All constraints slack implies all dual variables zero, implies dual value zero, implies primal value zero.)

8. Consider the following linear programming problem:

$$\max c \cdot x \text{ subject to } Ax \leq b, x \geq 0 \qquad (\mathbf{P})$$

and its Dual, **D**.

Suppose that there exists $\tilde{x}$ that feasible for **P**. That is, suppose $A\tilde{x} \leq b, \tilde{x} \geq 0$.

Which of the following statements are always true (when such an $\tilde{x}$ exists)? Indicate your answer by writing "True" or "False" following the statement.

(a) If $\tilde{y}$ is feasible for **D** then $b \cdot \tilde{y} \geq \tilde{y}A\tilde{x} \geq c \cdot \tilde{x}$.

Yes. This follows from the constraints (done in class).

(b) **D** is bounded.

Yes, by $c \cdot \tilde{x}$.

(c) The problem:

$$\max c \cdot x \text{ subject to } Ax \leq b/4, x \geq 0$$

is feasible.

Yes. $\tilde{x}/4$ satisfies the constraints.

(d) The linear programming problem obtained by multiplying all of the entries in the first column of A by the same positive constant (and not changing any other part of the problem) is feasible.

Yes. (Divide the first component of $\tilde{x}$ by the constant.)

(e) All feasible values of **D** are at least as great as $c \cdot \tilde{x}$.

Yes. This is part of the duality theorem.