Econ 172A - Slides from Lecture 13

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Announcements

- Problems on Branch and Bound: 2007: Homework 3, #1; Final: #5,6 2008: Homework 3 # 1, 2; Final: #4.
- Midterm: Graded. Information on Webpage. Note: Answer to Form C, 2(k) corrected on 11-13.
- Thursday's quiz on "old" material (sensitivity).
- If you have concerns about grading, pay attention to instructions on regrades. In particular, do not write on the exam.

Shortest Route Problem

- Given: locations, $0, 1, 2, \ldots T$.
- Location 0 is the "source" or starting point.
- Location T is the "sink" or target.
- The cost $c(i, j) \geq 0$ of going from .location i to location j.
- Cost may be infinite if it is impossible to go directly from i to j.
- There is no direction or order to the locations.
- It may be possible to go from 1 to 3 and also from 3 to 1 (and the costs may be different: $c(1, 3)$ need not be equal to $c(3, 1)$).

Algorithim

Step I Assign a permanent label of 0 to the source.

Step II Assign a temporary label of $c(0, i)$ to location i.

Step III Make the minimum temporary label a permanent label (if the minimum occurs at more than one location, then all relevant labels may become permanent). Denote by P the set of locations with permanent labels; denote the label of location i by $l(i)$ (this label may be temporary or permanent). If all locations have permanent labels, then stop.

Step IV Let each location that does not have a permanent label get a new temporary label according to the formula:

$$l(j) = \min_{i \in P} \left(l(i) + c(i,j) \right).$$

To compute $l(j)$ you need to compare only two things: the old temporary label and the cost of getting to j directly from the most recently labeled nodes.

Continued

Step V If the target has a permanent label, stop. This label is equal to the minimum cost. Otherwise, return to Step III.
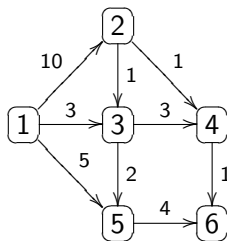
Intuition

- A node's label is an upper bound for the minimum cost of getting from the source to that node.
- A permanent label is guaranteed to be the true minimum cost of the shortest route (I prove this below).

Example

The figure (next page) represents a railroad network. The numbers beside each arc represent the time that it takes to move across the arc. Three locomotives are needed at point 6. There are two locomotives at point 2 and one at point 1. We can use the shortest route algorithm to find the routes that minimize the total time needed to move these locomotives to point 6.

The Network

We must solve two shortest route problems. The shortest route
from (1) to (6) and the shortest route from (2) to (6). To use the
algorithm, write temporary labels is the following table.

Iteration	1	2	3	4	5	6
1	0^*	10	3^{**}	∞	5	∞
2	0^*	10	3^*	6	5^{**}	∞
3	0^*	10	3^*	6^{**}	5_*	9
4	0^*	10	3^*	6^*	5^*	7^{**}
5	0^*	10^{**}	3^*	6^*	5^*	7^*

Interpretation

- Permanent labels are starred.
- The newest permanent label in each iteration has two stars.
- All other labels are temporary.

Discussion

- In the first iteration, the labels are just $c(0, j)$.
- In the second iteration, the minimum cost temporary label becomes permanent.
- The new temporary labels are computed using an extra possible route, namely the direct route from the new permanent label.
- For example, the label of location (6) decreases from 9 to 7 in Iteration 4 because at that point in the algorithm the possibility of getting from (6) from (4) becomes available.

Solution

- ▶ The shortest distance from (1) to (6) is 7.
- ▶ The route is found by working backwards.
- ▶ Look in the column corresponding to Node (6).
- ▶ Start at the bottom and move until the first time the cost changes.
- ▶ In this example, the cost changes from 7 to 9 between Iterations 3 and 4.

- ▶ Find the node that obtained its permanent label in Iteration 3 (or, generally, when the cost changes).
- ▶ In the example, this is Node (4).
- ▶ This means that the "last stop" before getting to Node (6) is Node (4).
- ▶ Continue to find where the route was immediately before Node (4).
- ▶ Do this by looking in the column corresponding to Node (4) and seeing the last time that the cost changed.
- ▶ In this case that last time the cost changed was between Iteration 1 and Iteration 2.
- ▶ It follows that the route stopped at Node (3) (which was permanently labeled in Iteration 1).

- Hence the shortest route must be: $(1) \to (3) \to (4) \to (6)$.

Another Application

A company needs to decide on a policy that determines when to replace machines.

The company has a T period planning horizon, and makes decisions on days $0, 1, \ldots T - 1$. The decision on date i is how long to keep a machine (until it is replaced).

Given information:

1. T
2. $c(i, j) \geq 0$ the cost of buying an item at time i and maintaining it until j.

Objective: Find a cost minimizing sequence of replacement dates. That is, find $t_1, t_2, dots, t_k$ to maximize

$$c(t_0, t_1) + c(t_1, t_2) + \ldots c(t_{k-1}, t_k)$$

subject to $0 < t_0 < t_1 < \ldots < t_k = T$.

This is the same as finding the shortest route from 0 to T.